Topological Defects and Phase Boundaries in Algebraic Conformal Quantum Field Theory*

Marcel Bischoff
http://www.theorie.physik.uni-goe.de/~bischoff

DFG Research Training Group 1493:
“Mathematical Structures in Modern Quantum Physics” – University of Göttingen

NCGOA 2014 Spring Institute ”Subfactors, CFT, and VOA” -
Vanderbilt University, May 8, 2014

*work in progress with R. Longo, Y. Kawahigashi and K.-H. Rehren
Introduction

- Algebraic quantum field theory: A family of algebras containing all local observables/operations associated with space-time regions.

- Conformal Quantum Field Theory (CQFT) in 1 and 2 dimension described by AQFT quite successful, e.g. partial classification results (e.g. $c < 1$) (Kawahigashi, Longo 2004).

- Topological Field Theory (TFT) construction of CFT on Riemann surfaces with boundaries/defects (Fuchs, Runkel, Schweigert (2002+)).
Warm up

Subfactor $N \subset M$, $[M : N] < \infty$, finite depth.

\[ M K_N := M L^2 M_N, \ N H_M \] its dual.

III $\iota : M \leftarrow N$ inclusion map, $\bar{\iota} : N \leftarrow M$ its dual.

Categories $N \mathcal{C}_N, M \mathcal{C}_N, N \mathcal{C}_M, M \mathcal{C}_M$ generated by

\[
\begin{align*}
\text{II} & \quad \text{Submodules } L \prec \cdots \Box K \Box H \Box K \Box \cdots, \\
& \quad \text{e.g. } M \mathcal{C}_N = \{ L \prec K \Box (H \Box K)^{\otimes n} \} \subset \text{Bim}(M, N). \\
\text{III} & \quad \text{Subsectors } \rho \prec \cdots \circ \iota \circ \bar{\iota} \circ \iota \circ \bar{\iota} \circ \cdots, \text{ e.g.} \\
& \quad M \mathcal{C}_N = \{ \rho \prec \iota \circ (\bar{\iota} \circ \iota)^{\circ n} \} \subset \text{Mor}(N, M).
\end{align*}
\]

Morita equivalence of fusion categories: $N \mathcal{C}_N \cong_{\text{Morita}} M \mathcal{C}_M$.

"Morita context" = 2-category:

- 0-cells: $\{ N, M \}$
- 1-cells: objects in $N \mathcal{C}_N, M \mathcal{C}_N, N \mathcal{C}_M, M \mathcal{C}_M$.
- 2-cells: intertwiners/bimodule maps
**Warm up**

Subfactor $N \subset M$, $[M : N] < \infty$, finite depth.

II $MK_N := M L^2 M_N$, $NH_M$ its dual.

III $\iota: M \leftarrow N$ inclusion map, $\bar{\iota}: N \leftarrow M$ its dual.

Categories $N C_N$, $M C_N$, $N C_M$, $M C_M$ generated by

II Submodules $L \prec \cdots \boxtimes K \boxtimes H \boxtimes K \boxtimes \cdots$,

e.g. $M C_N = \{L \prec K \boxtimes (H \boxtimes K)^{\otimes n}\} \subset \text{Bim}(M, N)$.

III Subsectors $\rho \prec \cdots \circ \iota \circ \bar{\iota} \circ \iota \circ \bar{\iota} \circ \cdots$,
e.g. $M C_N = \{\rho \prec \iota \circ (\bar{\iota} \circ \iota)^{\otimes n}\} \subset \text{Mor}(N, M)$.

**Morita equivalence of fusion categories:** $N C_N \approx_{\text{Morita}} M C_M$.

**“Morita context” = 2-category:**

- 0-cells: $\{N, M\}$
- 1-cells: objects in $N C_N$, $M C_N$, $N C_M$, $M C_M$.
- 2-cells: intertwiners/bimodule maps
Warm up

Subfactor $N \subset M$, $[M : N] < \infty$, finite depth.

1. $MK_N := ML^2M_N$, $NH_M$ its dual.

3. $\iota : M \leftarrow N$ inclusion map, $\bar{\iota} : N \leftarrow M$ its dual.

Categories $\mathcal{NC}_N, \mathcal{MC}_N, \mathcal{NC}_M, \mathcal{MC}_M$ generated by

2. Submodules $L \prec \cdots \boxtimes K \boxtimes H \boxtimes K \boxtimes \cdots$, e.g. $\mathcal{MC}_N = \{ L \prec K \boxtimes (H \boxtimes K)^{\otimes n} \} \subset \text{Bim}(M, N)$.

3. Subsectors $\rho \prec \cdots \circ \iota \circ \bar{\iota} \circ \iota \circ \bar{\iota} \circ \cdots$, e.g. $\mathcal{MC}_N = \{ \rho \prec \iota \circ (\bar{\iota} \circ \iota)^{\circ n} \} \subset \text{Mor}(N, M)$.

**Morita** equivalence of fusion categories: $\mathcal{NC}_N \approx_{\text{Morita}} \mathcal{MC}_M$.

“Morita context” = 2-category:

- 0-cells: $\{ N, M \}$
- 1-cells: objects in $\mathcal{NC}_N, \mathcal{MC}_N, \mathcal{NC}_M, \mathcal{MC}_M$.
- 2-cells: intertwiners/bimodule maps
Conversely starting with given (unitary) fusion category $\mathcal{C}_N$

1. $\mathcal{C}_N \subset \text{Bim}(N, N)$ (full and replete)
2. $\mathcal{C}_N \subset \text{End}(N)$ (full and replete)

consider subfactor (extension) $N \subset M$, such that:

1. $N L^2 M_N \equiv H \boxtimes K \cong N L^2 M_N \in \mathcal{C}_N$.
2. $\bar{\iota} \circ \iota \in \mathcal{C}_N$ (dual canonical endomorphism).

There are only finitely many such irreducible $N \subset M$ (up to equivalence).

If $\mathcal{C}_N$ has a braiding, i.e. is a (unitary) ribbon fusion category, we call the pair $(N \subset M, \mathcal{C}_N)$ a braided subfactor.

If further the braiding is non-degenerate, i.e. $\mathcal{C}_N$ is a (unitary) modular tensor category (UMTC), we call the pair $(N \subset M, \mathcal{C}_N)$ a non-degenerately braided subfactor.
Conversely starting with given (unitary) fusion category $\mathcal{C}_N$

\begin{itemize}
\item $\mathcal{C}_N \subset \text{Bim}(N,N)$ (full and replete)
\item $\mathcal{C}_N \subset \text{End}(N)$ (full and replete)
\end{itemize}

consider subfactor (extension) $N \subset M$, such that:

\begin{itemize}
\item $N L^2 M_N \equiv H \boxtimes K \cong N L^2 M_N \in \mathcal{C}_N$.
\item $\bar{\iota} \circ \iota \in \mathcal{C}_N$ (dual canonical endomorphism).
\end{itemize}

There are only finitely many such irreducible $N \subset M$ (up to equivalence).

If $\mathcal{C}_N$ has a braiding, i.e. is a (unitary) ribbon fusion category, we call the pair $(N \subset M, \mathcal{C}_N)$ a braided subfactor.

If further the braiding is non-degenerate, i.e. $\mathcal{C}_N$ is a (unitary) modular tensor category (UMTC), we call the pair $(N \subset M, \mathcal{C}_N)$ a non-degeneratly braided subfactor.
Conversely starting with given (unitary) fusion category $\mathcal{N}C_{\mathcal{N}}$

1. $\mathcal{N}C_{\mathcal{N}} \subset \text{Bim}(\mathcal{N}, \mathcal{N})$ (full and replete)
2. $\mathcal{N}C_{\mathcal{N}} \subset \text{End}(\mathcal{N})$ (full and replete)

consider subfactor (extension) $\mathcal{N} \subset \mathcal{M}$, such that:

1. $\mathcal{N}L^2\mathcal{M}_{\mathcal{N}} \equiv H \boxtimes K \cong \mathcal{N}L^2\mathcal{M}_{\mathcal{N}} \in \mathcal{N}C_{\mathcal{N}}$
2. $\bar{\iota} \circ \iota \in \mathcal{N}C_{\mathcal{N}}$ (dual canonical endomorphism).

There are only finitely many such irreducible $\mathcal{N} \subset \mathcal{M}$ (up to equivalence).

If $\mathcal{N}C_{\mathcal{N}}$ has a braiding, i.e. is a (unitary) ribbon fusion category, we call the pair $(\mathcal{N} \subset \mathcal{M}, \mathcal{N}C_{\mathcal{N}})$ a braided subfactor.

If further the braiding is non-degenerate, i.e. $\mathcal{N}C_{\mathcal{N}}$ is a (unitary) modular tensor category (UMTC), we call the pair $(\mathcal{N} \subset \mathcal{M}, \mathcal{N}C_{\mathcal{N}})$ a non-degenerately braided subfactor.
Conversely starting with given (unitary) fusion category $N\mathcal{C}_N$

1. $N\mathcal{C}_N \subset \text{Bim}(N, N)$ (full and replete)
2. $N\mathcal{C}_N \subset \text{End}(N)$ (full and replete)

consider subfactor (extension) $N \subset M$, such that:

1. $N L^2 M_N \equiv H \boxtimes K \cong N L^2 M_N \in N\mathcal{C}_N$.
2. $\bar{\iota} \circ \iota \in N\mathcal{C}_N$ (dual canonical endomorphism).

There are only finitely many such irreducible $N \subset M$ (up to equivalence).

If $N\mathcal{C}_N$ has a braiding, i.e. is a (unitary) ribbon fusion category, we call the pair $(N \subset M, N\mathcal{C}_N)$ a braided subfactor.

If further the braiding is non-degenerate, i.e. $N\mathcal{C}_N$ is a (unitary) modular tensor category (UMTC), we call the pair $(N \subset M, N\mathcal{C}_N)$ a non-degenerate braided subfactor.
Coordinate free description of "Ocneanu cells" = 2-category:

- **0-cells:** \{N, M\}
- **1-cells:** objects in $\mathcal{N}\mathcal{C}_N$ (UMTC), $\mathcal{M}\mathcal{C}_N$, $\mathcal{N}\mathcal{C}_M$, $\mathcal{M}\mathcal{C}_M$.
- **2-cells:** intertwiners/bimodule maps

What is the connection to algebraic conformal QFT? Briefly:

- $N$ = hyperfinite type III$_1$ factor prescribing a local algebra of observables $N = \mathcal{A}(I)$ of a fixed net $I \mapsto \mathcal{A}(I)$.
- $\mathcal{N}\mathcal{C}_N$ = UMTC of Doplicher–Haag–Roberts representations of $\mathcal{A}$ localized in $I$.
- $N \subset M$ (up to Morita equivalence) specifies a full CFT on Minkowski space (talk: Kawahigashi).
- $\mathcal{N}\mathcal{C}_M$ describes boundary conditions (talk: Kawahigashi).
- $\mathcal{M}\mathcal{C}_M$ fusion category of defects
- $\mathcal{M}_a\mathcal{C}_M$ boundaries between two full CFTs associated with $N \subset M_a, M_b$, respectively.

Marcel Bischoff (Uni Göttingen)  Defects and Boundaries in Algebraic Conformal QFT  Vanderbilt May 8, 2014
Coordinate free description of “Ocneanu cells” = 2-category:

- 0-cells: \( \{N, M\} \)
- 1-cells: objects in \( NCN \) (UMTC), \( MCN, NCM, MCM \).
- 2-cells: intertwiners/bimodule maps

What is the connection to algebraic conformal QFT? Briefly:

- \( N \) = hyperfinite type \( \text{III}_1 \) factor prescribing a local algebra of observables \( N = A(I) \) of a fixed net \( I \mapsto A(I) \).
- \( NCN \) = UMTC of Doplicher–Haag–Roberts representations of \( A \) localized in \( I \).
- \( N \subset M \) (up to Morita equivalence) specifies a full CFT on Minkowski space (talk: Kawahigashi).
- \( NCM \) describes boundary conditions (talk: Kawahigashi).
- \( MCM \) fusion category of defects
- \( MaCMb \) boundaries between two full CFTs associated with \( N \subset Ma, Mb \), respectively.
The 1-morphism $\bar{\rho} : N \to M$ is a **dual (conjugate)** of $\rho : M \to N$ if there exist intertwiners $R \in \text{Hom}(\text{id}_M, \bar{\rho}\rho)$ and $\bar{R} \in \text{Hom}(\text{id}_N, \rho\bar{\rho})$.

\[
R = \begin{array}{c}
\bar{\rho} \\
\rho \\
\text{id}_M
\end{array} \quad \bar{R} = \begin{array}{c}
\rho \\
\bar{\rho} \\
\text{id}_N
\end{array}
\]

standard solution of the **zig-zag identities**:

\[
\rho \\
\rho
\]

\[
= \begin{array}{c}
\rho \\
\rho
\end{array}
\]

\[
\bar{\rho} \\
\bar{\rho}
\]

\[
= \begin{array}{c}
\bar{\rho} \\
\bar{\rho}
\end{array}
\]

**Dimension** of an 1-morphism $= \text{square root of index of subfactor}$:

\[
d\rho = d\bar{\rho} = \begin{array}{c}
\quad \\
\quad
\end{array} = \begin{array}{c}
\quad \\
\quad
\end{array} = [M : \rho(N)]^{\frac{1}{2}}
\]
Subfactors and algebra objects

Subfactor $N \subset M$, $[M : N] < \infty$, fin. depth. Define triple $\Theta$ in $\text{End}(N)$

\[
\theta \theta \theta = \bar{\iota} \iota \iota,
\]

(algebra)

\[
\theta \theta = \bar{\iota} \iota,
\]

(unit)

\[
\theta \theta \theta = \bar{\iota} \iota \iota \iota.
\]

(co-multiplication)

Gives $\theta = \bar{\iota} \circ \iota$ structure of an (co-)algebra or Q-system:

\[
\theta \theta \theta = \bar{\iota} \iota \iota ; \theta \theta = \bar{\iota} \iota = \theta
\]

(co-associativity)

\[
\theta = \bar{\iota} \iota \iota \iota
\]

(unit law)
Subfactor $N \subset M$, $[M : N] < \infty$, fin. depth. Define triple $\Theta$ in $\text{End}(N)$

\[
\begin{align*}
\theta & = \bar{i} \circ i, \\
\theta & = \bar{i} \circ i, \\
\theta & = \bar{i} \circ i.
\end{align*}
\]

(algebra) \hspace{2cm} (unit) \hspace{2cm} (co-multiplication)

Gives $\theta = \bar{i} \circ i$ structure of an (co-)algebra or Q-system:

\[
\begin{align*}
\theta \theta \theta & = \theta \theta \theta; \\
\theta \theta & = \theta \theta; \\
\theta & = \theta.
\end{align*}
\]

(co-associativity) \hspace{2cm} (unit law)
Let $N$ be a type III factor (Longo 1994) (or use bimodules)

$$N \subset M \leftrightarrow \text{algebra } \Theta \text{ in } \text{End}(N)$$

Namely, the Q-system defines a conditional expectation $E$ on $N$ and $M$ is given by the Jones basic construction $E(N) \subset N \subset M$.

Fixing a URFC or UMTC $N\mathcal{C}_N \subset \text{End}(N)$

$$\text{Braided subfactor } N \subset M \text{ in } N\mathcal{C}_N \leftrightarrow \text{algebra } \Theta \text{ in } N\mathcal{C}_N$$

\begin{itemize}
  \item $N\mathcal{C}_M \cong \text{category of (right) } \Theta\text{-modules by tensoring on the right by } \iota.$
  \item $M_a\mathcal{C}_M b \cong \text{category of } \Theta_a\text{-}\Theta_b\text{ bimodules by } \beta \mapsto \iota_a \circ \beta \circ \iota_b.$
\end{itemize}
Let $N$ be a type III factor (Longo 1994) (or use bimodules)

\[ N \subset M \leftrightarrow \text{algebra } \Theta \text{ in } \text{End}(N) \]

Namely, the Q-system defines a conditional expectation $E$ on $N$ and $M$ is given by the Jones basic construction $E(N) \subset N \subset M$.

Fixing a URFC or UMTC $N\mathcal{C}_N \subset \text{End}(N)$

\[ \text{Braided subfactor } N \subset M \text{ in } N\mathcal{C}_N \leftrightarrow \text{algebra } \Theta \text{ in } N\mathcal{C}_N \]

- $N\mathcal{C}_M \cong$ category of (right) $\Theta$-modules by tensoring on the right by $\iota$.
- $M_a\mathcal{C}_M_b \cong$ category of $\Theta_a$-$\Theta_b$ bimodules by $\beta \mapsto \bar{i}_a \circ \beta \circ i_b$. 
Preliminaries on AQFT

Boundaries / Defects

Classification (modular case)
1+1D (local) conformal net $\mathcal{A}$ on Minkowski space $\mathbb{R}^{1,1}$:

$$\mathbb{R}^{1,1} \ni O \mapsto \mathcal{A}(O) \subset \mathcal{B}({\mathcal{H}_\mathcal{A}}), \quad \mathcal{H}_\mathcal{A} \text{ fixed Hilbert space}$$

1. Isotony: $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$

2. Locality: $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$ if $O_1$ and $O_1$ spacelike separated.

3. Covariance: $U_{\mathcal{A}}$ is a unitary **positive-energy** representation of the “2D conformal group”, s.t. $U_{\mathcal{A}}(g)\mathcal{A}(O)U_{\mathcal{A}}(g)^* = \mathcal{A}(gO)$.

4. Vacuum: $\exists \Omega$ is unique translation invariant vector.
Chiral (local) conformal net $\mathcal{A}$ on $S^1 \cong \overline{\mathbb{R}}$ (compactified light-ray):

$$S^1 \ni I \mapsto \mathcal{A}(I) \subset \mathcal{B}(\mathcal{H}_\mathcal{A}_0), \quad \mathcal{H}_\mathcal{A}_0 \text{ fixed Hilbert space}$$

1. Isotony: $I \subset J \Rightarrow \mathcal{A}_0(I) \subset \mathcal{A}_0(J)$
2. Locality: $[\mathcal{A}_0(I), \mathcal{A}_0(J)] = \{0\}$ if $I \cap J = \emptyset$.
3. Covariance: $U_{\mathcal{A}_0}$ is a unitary positive-energy representation of the Möbius group/diffeomorphism group, s.t.
   $$U_{\mathcal{A}_0}(g)\mathcal{A}_0(I)U_{\mathcal{A}_0}(g)^* = \mathcal{A}(gI).$$
4. Vacuum: $\exists \Omega$ is unique translation invariant vector.
Representation of $\mathcal{A} = \{\mathcal{A}(I)\}_{I \subset S^1}$ is a family:

$$\pi = \{\pi_I : \mathcal{A}(I) \to \mathcal{B}(\mathcal{H}_\pi)\},$$

which is compatible, i.e. $\pi_J \upharpoonright \mathcal{A}(I) = \pi_I$ for $I \subset J$.

- For all $I_0$ exists a $\rho \cong \pi$ on $\mathcal{H}$, s.t. $\rho_I = \text{id}_{\mathcal{A}(I)}$ for $I \cap I_0 = \emptyset$.
- $\rho_J$ is localized DHR endomorphism: $\rho_J(\mathcal{A}(J)) \subset \mathcal{A}(J)$ for all $J \supset I_0$.
- Tensor product: composition of localized endomorphisms.
- $\text{Rep}^I(\mathcal{A}) \subset \text{End}(\mathcal{A}(I))$ (full and replete).
- $\exists$ natural braiding (Fredenhagen, Rehren, Schroer (1989)).

Theorem ((Kawahigashi, Longo, Müger (2001)))

If $\mathcal{A}$ is a completely rational conformal net. Complete Rationality, $\implies$ $\text{Rep}(\mathcal{A})$ is a modular $\mathbb{C}^*$-tensor category / UMTC.
Complete Rationality and Modular Tensor Categories

**Representation** of \( A = \{A(I)\}_{I \subset S^1} \) is a family:

\[
\pi = \{\pi_I : A(I) \to B(H_\pi)\},
\]

which is compatible, i.e. \( \pi_J \upharpoonright A(I) = \pi_I \) for \( I \subset J \).

- For all \( I_0 \) exists a \( \rho \cong \pi \) on \( H \), s.t. \( \rho_I = \text{id}_{A(I)} \) for \( I \cap I_0 = \emptyset \).
- \( \rho_J \) is localized DHR endomorphism: \( \rho_J(A(J)) \subset A(J) \) for all \( J \supset I_0 \).
- Tensor product: composition of localized endomorphisms.
- \( \text{Rep}^I(A) \subset \text{End}(A(I)) \) (full and replete).
- \( \exists \) natural braiding (Fredenhagen, Rehren, Schroer (1989)).

**Theorem** ((Kawahigashi, Longo, Müger (2001)))

*If \( A \) is a completely rational conformal net. \( \Rightarrow \) \( \text{Rep}(A) \) is a modular C*-tensor category / UMTC.*
**Representation** of $\mathcal{A} = \{\mathcal{A}(I)\}_{I \subset S^1}$ is a family:

$$\pi = \{\pi_I : \mathcal{A}(I) \to \mathcal{B}(\mathcal{H}_\pi)\},$$

which is compatible, i.e. $\pi_J \upharpoonright \mathcal{A}(I) = \pi_I$ for $I \subset J$.

- For all $I_0$ exists a $\rho \cong \pi$ on $\mathcal{H}$, s.t. $\rho_I = \text{id}_{\mathcal{A}(I)}$ for $I \cap I_0 = \emptyset$.
- $\rho_J$ is localized DHR endomorphism: $\rho_J(\mathcal{A}(J)) \subset \mathcal{A}(J)$ for all $J \supset I_0$.
- Tensor product: composition of localized endomorphisms.
- $\text{Rep}^I(\mathcal{A}) \subset \text{End}(\mathcal{A}(I))$ (full and replete).
- $\exists$ natural braiding (Fredenhagen, Rehren, Schroer (1989)).

**Theorem ((Kawahigashi, Longo, Müger (2001)))**

*If $\mathcal{A}$ is a completely rational conformal net. Complete Rationality $\implies \text{Rep}(\mathcal{A})$ is a modular $C^*$-tensor category / UMTC.*
Complete Rationality and Modular Tensor Categories

**Representation** of $\mathcal{A} = \{ \mathcal{A}(I) \}_{I \subset S^1}$ is a family:

$$\pi = \{ \pi_I : \mathcal{A}(I) \to B(\mathcal{H}_\pi) \},$$

which is compatible, i.e. $\pi_J \upharpoonright \mathcal{A}(I) = \pi_I$ for $I \subset J$.

- For all $I_0$ exists a $\rho \cong \pi$ on $\mathcal{H}$, s.t. $\rho_I = \text{id}_{\mathcal{A}(I)}$ for $I \cap I_0 = \emptyset$.
- $\rho_J$ is localized DHR endomorphism: $\rho_J(\mathcal{A}(J)) \subset \mathcal{A}(J)$ for all $J \supset I_0$.
- Tensor product: composition of localized endomorphisms.
- $\text{Rep}^I(\mathcal{A}) \subset \text{End}(\mathcal{A}(I))$ (full and replete).
- $\exists$ natural braiding (Fredenhagen, Rehren, Schroer (1989)).

**Theorem** ((Kawahigashi, Longo, Müger (2001)))

*If $\mathcal{A}$ is a completely rational conformal net. Complete Rationality, $\implies \text{Rep}(\mathcal{A})$ is a modular $C^*$-tensor category / UMTC.*
Complete Rationality and Modular Tensor Categories

**Representation** of \( \mathcal{A} = \{ \mathcal{A}(I) \}_{I \subset S^1} \) is a family:

\[
\pi = \{ \pi_I: \mathcal{A}(I) \to B(\mathcal{H}_\pi) \}
\]

which is compatible, i.e. \( \pi_J | \mathcal{A}(I) = \pi_I \) for \( I \subset J \).

- For all \( I_0 \) exists a \( \rho \cong \pi \) on \( \mathcal{H} \), s.t. \( \rho_I = \text{id}_{\mathcal{A}(I)} \) for \( I \cap I_0 = \emptyset \).
- \( \rho_J \) is localized DHR endomorphism: \( \rho_J(\mathcal{A}(J)) \subset \mathcal{A}(J) \) for all \( J \supset I_0 \).
- Tensor product: composition of localized endomorphisms.
- \( \text{Rep}^I(\mathcal{A}) \subset \text{End}(\mathcal{A}(I)) \) (full and replete).
- \( \exists \) natural braiding (Fredenhagen, Rehren, Schroer (1989)).

**Theorem** ((Kawahigashi, Longo, M"uger (2001)))

If \( \mathcal{A} \) is a completely rational conformal net, \( \implies \text{Rep}(\mathcal{A}) \) is a modular C*-tensor category / UMTC.
**Representation of** $\mathcal{A} = \{\mathcal{A}(I)\}_{I \subset S^1}$ **is a family:**

$$\pi = \{\pi_I : \mathcal{A}(I) \to B(\mathcal{H}_\pi)\},$$

which is compatible, i.e. $\pi_J \upharpoonright \mathcal{A}(I) = \pi_I$ for $I \subset J$.

- For all $I_0$ exists a $\rho \cong \pi$ on $\mathcal{H}$, s.t. $\rho_I = \text{id}_{\mathcal{A}(I)}$ for $I \cap I_0 = \emptyset$.
- $\rho_J$ is localized DHR endomorphism: $\rho_J(\mathcal{A}(J)) \subset \mathcal{A}(J)$ for all $J \supset I_0$.
- Tensor product: composition of localized endomorphisms.
- $\text{Rep}^I(\mathcal{A}) \subset \text{End}(\mathcal{A}(I))$ (full and replete).
- $\exists$ natural braiding (Fredenhagen, Rehren, Schroer (1989)).

**Theorem ((Kawahigashi, Longo, Müger (2001)))**

*If $\mathcal{A}$ is a completely rational conformal net.*  

$\implies \text{Rep}(\mathcal{A})$ is a modular C*-tensor category / UMTC.
Representation of $\mathcal{A} = \{\mathcal{A}(I)\}_{I \subset \mathbb{S}^1}$ is a family:

$$\pi = \{\pi_I : \mathcal{A}(I) \to \mathcal{B}(\mathcal{H}_\pi)\},$$

which is compatible, i.e. $\pi_J|_{\mathcal{A}(I)} = \pi_I$ for $I \subset J$.

- For all $I_0$ exists a $\rho \cong \pi$ on $\mathcal{H}$, s.t. $\rho_I = \text{id}_{\mathcal{A}(I)}$ for $I \cap I_0 = \emptyset$.
- $\rho_J$ is localized DHR endomorphism: $\rho_J(\mathcal{A}(J)) \subset \mathcal{A}(J)$ for all $J \supset I_0$.
- Tensor product: composition of localized endomorphisms.
- $\text{Rep}^I(\mathcal{A}) \subset \text{End}(\mathcal{A}(I))$ (full and replete).
- $\exists$ natural braiding (Fredenhagen, Rehren, Schroer (1989)).

Theorem ((Kawahigashi, Longo, Müger (2001)))

If $\mathcal{A}$ is a completely rational conformal net. \(\Rightarrow\) $\text{Rep}(\mathcal{A})$ is a modular $\mathcal{C}^*$-tensor category / UMTC.
Complete Rationality and Modular Tensor Categories

Example

- Conformal nets with central charge $c < 1$.
- Conformal nets associated with even lattices (Dong, Xu 2006).
- Conformal nets associated with PER of loop groups (SU($N$) (Wassermann 1998, Xu 2000), simply-laced groups level 1):
  \[
  A_{G,k}(I) = \pi(LIG)^{''}
  \]
- Finite index extensions (e.g. simple currents) and finite index subnets (e.g. finite orbifolds) of completely rational conformal nets.

Conjecture

Every unitary, $C_2$-cofinite VOA (+add. assumptions) gives a completely rational conformal net on $S^1$ and the associated MTCs of representations are equivalent.
Complete Rationality and Modular Tensor Categories

Example

- Conformal nets with central charge $c < 1$.
- Conformal nets associated with even lattices (Dong, Xu 2006).
- Conformal nets associated with PER of loop groups ($\text{SU}(N)$ (Wassermann 1998, Xu 2000), simply-laced groups level 1):

$$A_{G,k}(I) = \pi(LIG)'$$

- Finite index extensions (e.g. simple currents) and finite index subnets (e.g. finite orbifolds) of completely rational conformal nets.

Conjecture

Every unitary, $C_2$-cofinite VOA (+add. assumptions) gives a completely rational conformal net on $S^1$ and the associated MTCs of representations are equivalent.
Local and (non-local) extensions

\( \mathcal{B} \supset \mathcal{A} \) irreducible local extension with finite index \( [\mathcal{B}(I) : \mathcal{A}(I)] < \infty \), then the algebra/Q-system \( \Theta \) associated with \( \mathcal{A}(I) \subset \mathcal{B}(I) \) is in \( \text{Rep}^I(\mathcal{A}) \) and commutative, also the converse holds:

\[
\begin{array}{c}
\mathcal{B} \supset \mathcal{A} \text{ finite local} \\
\longleftrightarrow \\
\text{commutative algebra } \Theta \text{ in } \text{Rep}^I(\mathcal{A})
\end{array}
\]

\( \Theta \) commutative = quantum subgroups, classifications known:

Example

- The quantum subgroups SU(2)_k are in correspondence with \( A_n \) (=trivial extension), \( D_{\text{even}} \) (simple currents), \( E_{6,8} \) Dynkin diagrams.
- The quantum subgroups of Vir_c for \( c < 1 \) with certain pairs of such Dynkin diagrams.

(Non-local) but relatively local extensions:

\[
\begin{array}{c}
\mathcal{B} \supset \mathcal{A} \text{ finite (non-local)} \\
\longleftrightarrow \\
\text{algebra } \Theta \text{ in } \text{Rep}^I(\mathcal{A})
\end{array}
\]
Local and (non-local) extensions

$\mathcal{B} \supset \mathcal{A}$ irreducible local extension with finite index $[\mathcal{B}(I) : \mathcal{A}(I)] < \infty$, then the algebra/Q-system $\Theta$ associated with $\mathcal{A}(I) \subset \mathcal{B}(I)$ is in $\text{Rep}^I(\mathcal{A})$ and commutative, also the converse holds:

$\mathcal{B} \supset \mathcal{A}$ finite local $\iff$ commutative algebra $\Theta$ in $\text{Rep}^I(\mathcal{A})$

$\Theta$ commutative = quantum subgroups, classifications known:

Example

- The quantum subgroups $\text{SU}(2)_k$ are in correspondence with $A_n$ (=trivial extension), $D_{\text{even}}$ (simple currents), $E_{6,8}$ Dynkin diagrams.
- The quantum subgroups of $\text{Vir}_c$ for $c < 1$ with certain pairs of such Dynkin diagrams.

(Non-local) but relatively local extensions:

$\mathcal{B} \supset \mathcal{A}$ finite (non-local) $\iff$ algebra $\Theta$ in $\text{Rep}^I(\mathcal{A})$
Local and (non-local) extensions

\[ \mathcal{B} \supset \mathcal{A} \] irreducible local extension with finite index \([\mathcal{B}(I) : \mathcal{A}(I)] < \infty\), then the algebra/Q-system \(\Theta\) associated with \(\mathcal{A}(I) \subset \mathcal{B}(I)\) is in \(\text{Rep}^I(\mathcal{A})\) and commutative, also the converse holds:

\[
\begin{array}{c}
\mathcal{B} \supset \mathcal{A} \text{ finite local} \iff \text{commutative algebra } \Theta \text{ in } \text{Rep}^I(\mathcal{A})
\end{array}
\]

\(\Theta\) commutative = **quantum subgroups**, classifications known:

**Example**

- The quantum subgroups \(SU(2)_k\) are in correspondence with \(A_n\) (=trivial extension), \(D_{\text{even}}\) (simple currents), \(E_{6,8}\) Dynkin diagrams.
- The quantum subgroups of \(\text{Vir}_c\) for \(c < 1\) with certain pairs of such Dynkin diagrams.

(Non-local) but relatively local extensions:

\[
\begin{array}{c}
\mathcal{B} \supset \mathcal{A} \text{ finite (non-local)} \iff \text{algebra } \Theta \text{ in } \text{Rep}^I(\mathcal{A})
\end{array}
\]
Local and (non-local) extensions

$\mathcal{B} \supset \mathcal{A}$ irreducible local extension with finite index $[\mathcal{B}(I) : \mathcal{A}(I)] < \infty$, then the algebra/Q-system $\Theta$ associated with $\mathcal{A}(I) \subset \mathcal{B}(I)$ is in $\text{Rep}^I(\mathcal{A})$ and commutative, also the converse holds:

$\mathcal{B} \supset \mathcal{A}$ finite local $\iff$ commutative algebra $\Theta$ in $\text{Rep}^I(\mathcal{A})$

$\Theta$ commutative $=$ quantum subgroups, classifications known:

Example

- The quantum subgroups $SU(2)_k$ are in correspondence with $A_n$ ($=$trivial extension), $D_{\text{even}}$ (simple currents), $E_{6,8}$ Dynkin diagrams.
- The quantum subgroups of $\text{Vir}_c$ for $c < 1$ with certain pairs of such Dynkin diagrams.

(Non-local) but relatively local extensions:

$\mathcal{B} \supset \mathcal{A}$ finite (non-local) $\iff$ algebra $\Theta$ in $\text{Rep}^I(\mathcal{A})$
One can define a \textbf{1+1D conformal net} on Minkowski space by

$$\mathcal{A}(O) = \mathcal{A}_+(I_1) \otimes \mathcal{A}_-(I_2) \subset B(\mathcal{H}_{\mathcal{A}_+} \otimes \mathcal{H}_{\mathcal{A}_-}),$$

where $\mathcal{A}_\pm$ are conformal nets on $\mathbb{R}$.

General 1+1D conformal nets are given by irreducible local extensions

$$\mathcal{B}(O) \supset \mathcal{A}(O) \equiv \mathcal{A}_+(I_1) \otimes \mathcal{A}_-(I_2).$$

Physically: Net $\mathcal{A} \equiv \mathcal{A}_+ \otimes \mathcal{A}_-$ describes \textbf{symmetries}, e.g. local conformal transformations ($\text{Vir}$), gauge transformations ($\text{L}_G$) etc.
One can define a 1+1D conformal net on Minkowski space by

$$\mathcal{A}(O) = \mathcal{A}_+(I_1) \otimes \mathcal{A}_-(I_2) \subset \mathcal{B}(\mathcal{H}_{\mathcal{A}_+} \otimes \mathcal{H}_{\mathcal{A}_-}),$$

where $\mathcal{A}_\pm$ are conformal nets on $\mathbb{R}$.

General 1+1D conformal nets are given by irreducible local extensions

$$\mathcal{B}(O) \supset \mathcal{A}(O) \equiv \mathcal{A}_+(I_1) \otimes \mathcal{A}_-(I_2).$$

Physically: Net $\mathcal{A} \equiv \mathcal{A}_+ \otimes \mathcal{A}_-$ describes symmetries, e.g. local conformal transformations (Vir), gauge transformations (LG) etc.
One can define a 1+1D conformal net on Minkowski space by

\[ A(O) = A_+(I_1) \otimes A_-(I_2) \subset B(\mathcal{H}_{A_+} \otimes \mathcal{H}_{A_-}), \]

where \( A_\pm \) are conformal nets on \( \mathbb{R} \).

General 1+1D conformal nets are given by irreducible local extensions

\[ B(O) \supset A(O) \equiv A_+(I_1) \otimes A_-(I_2). \]

\[ \mathcal{B} \supset A_+ \otimes A_- \text{ finite} \iff \text{com. algebra } \Theta \text{ in } \text{Rep}^I(A_+) \boxtimes \text{Rep}^J(A_-). \]

Physically: Net \( A \equiv A_+ \otimes A_- \) describes symmetries, e.g. local conformal transformations (Vir), gauge transformations (LG) etc.
Outline

Preliminaries on AQFT

Boundaries / Defects

Classification (modular case)
Local boundaries

Left observables $\mathcal{B}_L \supset \mathcal{A}$.

- Boundary invisible for $\mathcal{A} \equiv \mathcal{A}_+ \otimes \mathcal{A}_-$, $\mathcal{D}(O) := \mathcal{B}_L(O) \vee \mathcal{B}_R(O)$.

Right observables $\mathcal{B}_R \supset \mathcal{A}$.

- Locality: $[\mathcal{B}_L(O_L), \mathcal{B}_R(O_R)] = \{0\}$ for $O_L$ spacelike left of $O_R$.
- $\mathcal{B}_L(O_L) \subset \mathcal{B}_R(O_L^\leq)' \subset \mathcal{D}(O_L^\leq)'$.
- $\mathcal{B}_R(O_R) \subset \mathcal{B}_L(O_R^\geq)' \subset \mathcal{D}(O_R^\geq)'$. 
Local boundaries

**Left observables** $\mathcal{B}_L \supset \mathcal{A}$.

- Boundary invisible for $\mathcal{A} \equiv \mathcal{A}_+ \otimes \mathcal{A}_-$, $\mathcal{D}(O) := \mathcal{B}_L(O) \lor \mathcal{B}_R(O)$.

**Right observables** $\mathcal{B}_R \supset \mathcal{A}$.

- Locality: $[\mathcal{B}_L(O_L), \mathcal{B}_R(O_R)] = \{0\}$ for $O_L$ spacelike left of $O_R$.
- $\mathcal{B}_L(O_L) \subset \mathcal{B}_R(O_L^\leq)' \subset \mathcal{D}(O_L^\leq)'$.
- $\mathcal{B}_R(O_R) \subset \mathcal{B}_L(O_R^\geq)' \subset \mathcal{D}(O_R^\geq)'$.
Local boundaries

Left observables $\mathcal{B}_L \supset \mathcal{A}$.

- Boundary invisible for $\mathcal{A} \equiv \mathcal{A}_+ \otimes \mathcal{A}_-$, $\mathcal{D}(O) := \mathcal{B}_L(O) \lor \mathcal{B}_R(O)$.

Right observables $\mathcal{B}_R \supset \mathcal{A}$.

- Locality: $[\mathcal{B}_L(O_L), \mathcal{B}_R(O_R)] = \{0\}$ for $O_L$ spacelike left of $O_R$. 
- $\mathcal{B}_L(O_L) \subset \mathcal{B}_R(O_L^<)' \subset \mathcal{D}(O_L^<)'$. 
- $\mathcal{B}_R(O_R) \subset \mathcal{B}_L(O_R^>)' \subset \mathcal{D}(O_R^>)'$. 

Marcel Bischoff (Uni Göttingen)  
Defects and Boundaries in Algebraic Conformal QFT  
Vanderbilt May 8, 2014
Local boundaries

Left observables $\mathcal{B}_L \supset \mathcal{A}$.
- Boundary invisible for $\mathcal{A} \equiv \mathcal{A}_+ \otimes \mathcal{A}_-$, $\mathcal{D}(O) := \mathcal{B}_L(O) \lor \mathcal{B}_R(O)$.

Right observables $\mathcal{B}_R \supset \mathcal{A}$.

- Locality: $[\mathcal{B}_L(O_L), \mathcal{B}_R(O_R)] = \{0\}$ for $O_L$ spacelike left of $O_R$.  
- $\mathcal{B}_L(O_L) \subset \mathcal{B}_R(O_L^\prec)' \subset \mathcal{D}(O_L^\prec)'$.
- $\mathcal{B}_R(O_R) \subset \mathcal{B}_L(O_R^\succ)' \subset \mathcal{D}(O_R^\succ)'$. 
Local boundaries

Left observables $\mathcal{B}_L \supset \mathcal{A}$.

- Boundary invisible for $\mathcal{A} \equiv \mathcal{A}_+ \otimes \mathcal{A}_-$, $\mathcal{D}(O) := \mathcal{B}_L(O) \vee \mathcal{B}_R(O)$.

- Local observables: $[\mathcal{B}_L(O_L), \mathcal{B}_R(O_R)] = \{0\}$ for $O_L$ spacelike left of $O_R$.

- $\mathcal{B}_L(O_L) \subset \mathcal{B}_R(O_L^\leq)' \subset \mathcal{D}(O_L^\leq)'$.

- $\mathcal{B}_R(O_R) \subset \mathcal{B}_L(O_R^\geq)' \subset \mathcal{D}(O_R^\geq)'$. 
We fix

- (Chiral) symmetries $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$ with $\mathcal{A}_\pm$ local nets on $\mathbb{R}$.
- Left observables: local extension $\mathcal{B}_L \supset \mathcal{A}$.
- Right observables: local extension $\mathcal{B}_R \supset \mathcal{A}$.

$\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{B}_R$-boundary $\mathcal{D}$ is a (non-local) extension $\mathcal{D} \supset \mathcal{A}$ such that:

- left center of $\mathcal{D}$:
  $$\mathcal{C}_L(O) := \mathcal{D}(O) \cap \mathcal{D}(O^<)'$$

- right center of $\mathcal{D}$:
  $$\mathcal{C}_R(O) := \mathcal{D}(O) \cap \mathcal{D}(O^>)'$$
We fix

- (Chiral) symmetries $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$ with $\mathcal{A}_\pm$ local nets on $\mathbb{R}$.
- Left observables: local extension $\mathcal{B}_L \supset \mathcal{A}$.
- Right observables: local extension $\mathcal{B}_R \supset \mathcal{A}$.

$\mathcal{A}$–topological $\mathcal{B}_L–\mathcal{B}_R$-boundary $\mathcal{D}$ is a (non-local) extension $\mathcal{D} \supset \mathcal{A}$ such that:

left center of $\mathcal{D}$ : $\mathcal{C}_L(O) := \mathcal{D}(O) \cap \mathcal{D}(O^<)'$
right center of $\mathcal{D}$ : $\mathcal{C}_R(O) := \mathcal{D}(O) \cap \mathcal{D}(O^>)'$
Consider $\iota(N) \subset M$ with dual canonical endomorphism:

$$\theta = \bar{\iota} \circ \iota = \bigoplus_{\rho} n_{\rho} \cdot \rho .$$

- $M = N \lor \{ \mathcal{H}_{\rho}\}_{\rho} \equiv N \lor \{ \psi_{\rho,i}\}$,
- $\mathcal{H}_{\rho} := \text{Hom}(\iota, \iota\rho)$ with ONB $\{ \psi_{\rho,i}\}_{i=1}^{n_{\rho}}$.
- $\psi_{\rho,i}$ are charged intertwiners:

$$\psi_{\rho,i} \cdot \iota(n) = \iota(\rho(n)) \cdot \psi_{\rho,i} \quad n \in N .$$

- Unique “Fourier decomposition”:

$$m = \sum_{\rho,i} n_{\rho,i} \cdot \psi_{\rho,i} \quad n_{\rho,i} \in N .$$
Universal $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{B}_R$ boundary $\mathcal{D}_{\text{univ}}$, with no further relations between $\mathcal{B}_L$ and $\mathcal{B}_R$ besides locality:

- $\mathcal{B}_L(O) = \mathcal{A}(O) \lor \{\psi^O_{L,i}\}_i$
- $\mathcal{B}_R(O) = \mathcal{A}(O) \lor \{\psi^O_{R,j}\}_j$
- $\mathcal{D}_{\text{univ}}(O) = \mathcal{A}(O) \lor \{\psi^O_{L,i}\}_i \lor \{\psi^O_{R,j}\}_j$
- $\{\psi^O_{L,i}\}_i$ and $\{\psi^O_{R,j}\}_j$ fulfilling left-right locality:

$$[\psi^O_{L,i}, \psi^O_{R,j}] = 0 \quad \text{for all } i, j \text{ and } O_L \text{ space-like left of } O_R$$

**Theorem**

If $\mathcal{B}_L \supset \mathcal{A}$ and $\mathcal{B}_R \supset \mathcal{A}$ are local, irreducible, finite index extensions then there exists a unique $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{B}_R$ boundary $\mathcal{D}_{\text{univ}}$ with the above properties.
Universal $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{B}_R$ boundary $\mathcal{D}_{\text{univ}}$, with no further relations between $\mathcal{B}_L$ and $\mathcal{B}_R$ besides locality:

- $\mathcal{B}_L(O) = \mathcal{A}(O) \lor \{\psi^O_{L,i}\}_i$
- $\mathcal{B}_R(O) = \mathcal{A}(O) \lor \{\psi^O_{R,j}\}_j$
- $\mathcal{D}_{\text{univ}}(O) = \mathcal{A}(O) \lor \{\psi^O_{L,i}\}_i \lor \{\psi^O_{R,j}\}_j$
- $\{\psi^O_{L,i}\}_i$ and $\{\psi^O_{R,j}\}_j$ fulfilling left-right locality:

$$[\psi^O_{L,i}, \psi^O_{R,j}] = 0$$

for all $i, j$ and $O_L$ space-like left of $O_R$

**Theorem**

*If $\mathcal{B}_L \supset \mathcal{A}$ and $\mathcal{B}_R \supset \mathcal{A}$ are local, irreducible, finite index extensions then there exists a unique $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{B}_R$ boundary $\mathcal{D}_{\text{univ}}$ with the above properties.*
Morally: \( \mathcal{D}_{\text{univ}} \) is the "fusion" of
- \( \mathcal{B}_L \) as \( \mathcal{A} \)-topological \( \mathcal{B}_L - \mathcal{A} \) boundary and
- \( \mathcal{B}_R \) as \( \mathcal{A} \)-topological \( \mathcal{A} - \mathcal{B}_R \) boundary

over \( \mathcal{A} \):

\[
\mathcal{B}_L [\mathcal{D}_{\text{univ}}] \mathcal{B}_R = \mathcal{B}_L [\mathcal{B}_L] \mathcal{A} \boxtimes_{\mathcal{A}} \mathcal{A}[\mathcal{B}_R] \mathcal{B}_R.
\]

Mathematically: \( \mathcal{D}_{\text{univ}} \) is obtained by a braided product of Q-systems specifying the inclusions \( \mathcal{A} \subset \mathcal{B}_L \) and \( \mathcal{A} \subset \mathcal{B}_R \), respectively.

Result

\( \mathcal{D}_{\text{univ}} (O) \) are general not factors, i.e. \( \mathcal{D}_{\text{univ}} \) is reducible. The center \( \mathcal{D}_{\text{univ}} (O) \cap \mathcal{D}_{\text{univ}} (O)' \) is a finite algebra and the central decomposition of \( \mathcal{D}_{\text{univ}} \) gives “all” irreducible \( \mathcal{A} \)-topological \( \mathcal{B}_L - \mathcal{B}_R \) boundaries/defects.

- Decomposition

\[
\mathcal{D}_{\text{univ}} (O) \equiv \mathcal{A}(O) \lor \{\psi^O_{L,i}\}_i \lor \{\psi^O_{R,j}\}_j = \bigoplus_m \mathcal{D}_m (O).
\]

In \( \mathcal{D}_m (O) \) \exists \text{ relations} between \( \{\psi^O_{L,i}\}_i \) and \( \{\psi^O_{R,j}\}_j \).
**Universal construction II**

**Morally:** $\mathcal{D}_{\text{univ}}$ is the “fusion” of

- $\mathcal{B}_L$ as $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{A}$ boundary and
- $\mathcal{B}_R$ as $\mathcal{A}$–topological $\mathcal{A}$–$\mathcal{B}_R$ boundary

over $\mathcal{A}$:

$$\mathcal{B}_L[\mathcal{D}_{\text{univ}}]\mathcal{B}_R = \mathcal{B}_L[\mathcal{B}_L]\mathcal{A} \bigotimes_{\mathcal{A}} \mathcal{A}[\mathcal{B}_R]\mathcal{B}_R.$$  

**Mathematically:** $\mathcal{D}_{\text{univ}}$ is obtained by a braided product of Q-systems specifying the inclusions $\mathcal{A} \subset \mathcal{B}_L$ and $\mathcal{A} \subset \mathcal{B}_R$, respectively.

---

**Result**

$\mathcal{D}_{\text{univ}}(O)$ are general not factors, i.e. $\mathcal{D}_{\text{univ}}$ is reducible. The center $\mathcal{D}_{\text{univ}}(O) \cap \mathcal{D}_{\text{univ}}(O)'$ is a finite algebra and the central decomposition of $\mathcal{D}_{\text{univ}}$ gives “all” irreducible $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{B}_R$ boundaries/defects.

- **Decomposition**

  $$\mathcal{D}_{\text{univ}}(O) \equiv \mathcal{A}(O) \lor \{\psi^O_{\mathcal{L},i}\}_i \lor \{\psi^O_{\mathcal{R},j}\}_j = \bigoplus_m \mathcal{D}_m(O).$$

- In $\mathcal{D}_m(O)$ there are relations between $\{\psi^O_{\mathcal{L},i}\}_i$ and $\{\psi^O_{\mathcal{R},j}\}_j$. 

---

*Marcel Bischoff (Uni Göttingen)*

Defects and Boundaries in Algebraic Conformal QFT

Vanderbilt May 8, 2014
Morally: $\mathcal{D}_{\text{univ}}$ is the “fusion” of

- $\mathcal{B}_L$ as $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{A}$ boundary and
- $\mathcal{B}_R$ as $\mathcal{A}$–topological $\mathcal{A}$–$\mathcal{B}_R$ boundary

over $\mathcal{A}$:

$$\mathcal{B}_L[\mathcal{D}_{\text{univ}}]\mathcal{B}_R = \mathcal{B}_L[\mathcal{B}_L]\mathcal{A} \boxtimes_{\mathcal{A}} \mathcal{A}[\mathcal{B}_R]\mathcal{B}_R.$$ 

Mathematically: $\mathcal{D}_{\text{univ}}$ is obtained by a braided product of $Q$-systems specifying the inclusions $\mathcal{A} \subset \mathcal{B}_L$ and $\mathcal{A} \subset \mathcal{B}_R$, respectively.

Result

$\mathcal{D}_{\text{univ}}(O)$ are general not factors, i.e. $\mathcal{D}_{\text{univ}}$ is reducible. The center $\mathcal{D}_{\text{univ}}(O) \cap \mathcal{D}_{\text{univ}}(O)'$ is a finite algebra and the central decomposition of $\mathcal{D}_{\text{univ}}$ gives “all” irreducible $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{B}_R$ boundaries/defects.

- **Decomposition**

  $$\mathcal{D}_{\text{univ}}(O) \equiv \mathcal{A}(O) \vee \{\psi^O_{L,i}\}_i \vee \{\psi^O_{R,j}\}_j = \bigoplus_m \mathcal{D}_m(O).$$

- In $\mathcal{D}_m(O)$ \textbf{exists relations} between $\{\psi^O_{L,i}\}_i$ and $\{\psi^O_{R,j}\}_j$. 
Example: Ising model

**Virasoro net** $\mathcal{A}_0 = \text{Vir}_{1/2}$:
- $\text{Vir}_{c=1/2}$ (= even part of the free real Fermion net).
- Sectors: [id] (vacuum), $[\varepsilon]$ ($h = 1/2$), $[\sigma]$ ($h = 1/16$).
- Fusion rules: $\varepsilon \circ \varepsilon \cong \text{id}$, $\varepsilon \circ \sigma \cong \sigma$, $\sigma \circ \sigma \cong \text{id} \oplus \varepsilon$

Consider **Ising model** $\mathcal{B}_L = \mathcal{B}_R = \mathcal{B}_{LR}$, where
- $\mathcal{B}_{LR}(O) = \text{Vir}_{1/2}(I) \otimes \text{Vir}_{1/2}(J) \vee \{\Psi_\varepsilon, \Psi_\sigma\}$, $\mathcal{H}_{LR} = \bigoplus_{\rho=0,\varepsilon,\sigma} \mathcal{H}_\rho \otimes \mathcal{H}_\rho$
- Charged intertwinners (Cardy case bulk fields) of $\mathcal{B}_{LR}$:
  $\Psi_\varepsilon : \iota \to \iota \circ (\varepsilon \otimes \varepsilon)$, $\Psi_\sigma : \iota \to \iota \circ (\sigma \otimes \sigma)$.

**Irreducible topological** $\mathcal{B}_{LR}$–$\mathcal{B}_{LR}$-boundaries/defects:
- 1     invisible  $\Psi^L_\varepsilon = \Psi^R_\varepsilon$  $\Psi^L_\sigma = \Psi^R_\sigma$  $\Psi^L_\varepsilon = -\Psi^R_\varepsilon$  $\Psi^L_\sigma = -\Psi^R_\sigma$
- $\varepsilon$ spin-flip  $\Psi^L_\varepsilon = \Psi^R_\varepsilon$  $\Psi^L_\sigma = -\Psi^R_\sigma$
- $\sigma$ order-disorder  $\Psi^L_\varepsilon = -\Psi^R_\varepsilon$  $\Psi^L_\sigma \neq \Psi^R_\sigma$
Example: Ising model

**Virasoro net** $\mathcal{A}_0 = \text{Vir}_{1/2}$:

- $\text{Vir}_{c=1/2}$ (= even part of the free real Fermion net).
- Sectors: $[\text{id}]$ (vacuum), $[\varepsilon]$ ($h = 1/2$), $[\sigma]$ ($h = 1/16$).
- Fusion rules: $\varepsilon \circ \varepsilon \cong \text{id}$, $\varepsilon \circ \sigma \cong \sigma$, $\sigma \circ \sigma \cong \text{id} \oplus \varepsilon$

Consider **Ising model** $\mathcal{B}_L = \mathcal{B}_R = \mathcal{B}_{LR}$, where

- $\mathcal{B}_{LR}(O) = \text{Vir}_{1/2}(I) \otimes \text{Vir}_{1/2}(J) \vee \{\Psi_\varepsilon, \Psi_\sigma\}$, $\mathcal{H}_{LR} = \bigoplus_{\rho=0,\varepsilon,\sigma} \mathcal{H}_\rho \otimes \mathcal{H}_\rho$
- Charged intertwinners (Cardy case bulk fields) of $\mathcal{B}_{LR}$:
  $\Psi_\varepsilon : i \to i \circ (\varepsilon \otimes \varepsilon)$, $\Psi_\sigma : i \to i \circ (\sigma \otimes \sigma)$.

**Irreducible topological** $\mathcal{B}_{LR} - \mathcal{B}_{LR}$-boundaries/defects:

- **invisible** $\Psi^L_\varepsilon = \Psi^R_\varepsilon$ $\Psi^L_\sigma = \Psi^R_\sigma$
- **spin-flip** $\Psi^L_\varepsilon = \Psi^R_\varepsilon$ $\Psi^L_\sigma = -\Psi^R_\sigma$
- **order-disorder** $\Psi^L_\varepsilon = -\Psi^R_\varepsilon$ $\Psi^L_\sigma \neq \Psi^R_\sigma$
**Virasoro net** $\mathcal{A}_0 = \text{Vir}_{\frac{1}{2}}$:

- $\text{Vir}_{c=\frac{1}{2}}$ (= even part of the free real Fermion net).
- Sectors: $[\text{id}]$ (vacuum), $[\varepsilon]$ ($h = 1/2$), $[\sigma]$ ($h = 1/16$).
- Fusion rules: $\varepsilon \circ \varepsilon \cong \text{id}$, $\varepsilon \circ \sigma \cong \sigma$, $\sigma \circ \sigma \cong \text{id} \oplus \varepsilon$

Consider **Ising model** $\mathcal{B}_L = \mathcal{B}_R = \mathcal{B}_{\text{LR}}$, where

- $\mathcal{B}_{\text{LR}}(O) = \text{Vir}_{\frac{1}{2}}(I) \otimes \text{Vir}_{\frac{1}{2}}(J) \vee \{\Psi_{\varepsilon}, \Psi_{\sigma}\}$, $\mathcal{H}_{\text{LR}} = \bigoplus_{\rho=0,\varepsilon,\sigma} \mathcal{H}_\rho \otimes \mathcal{H}_\rho$
- Charged intertwinners (Cardy case bulk fields) of $\mathcal{B}_{\text{LR}}$:
  $\Psi_{\varepsilon} : \iota \rightarrow \iota \circ (\varepsilon \otimes \varepsilon)$, $\Psi_{\sigma} : \iota \rightarrow \iota \circ (\sigma \otimes \sigma)$.

**Irreducible topological** $\mathcal{B}_{\text{LR}} - \mathcal{B}_{\text{LR}}$-boundaries/defects:

1. **invisible**  $\Psi^L_{\varepsilon} = \Psi^R_{\varepsilon}$  $\Psi^L_{\sigma} = \Psi^R_{\sigma}$

2. **spin-flip**  $\Psi^L_{\varepsilon} = \Psi^R_{\varepsilon}$  $\Psi^L_{\sigma} = -\Psi^R_{\sigma}$

3. **order-disorder**  $\Psi^L_{\varepsilon} = -\Psi^R_{\varepsilon}$  $\Psi^L_{\sigma} \neq \Psi^R_{\sigma}$
Preliminaries on AQFT

Boundaries / Defects

Classification (modular case)
Characterization maximal local extensions

We assume: $\mathcal{A}_+ = \mathcal{A}_- := \mathcal{A}_0$ completely rational. Further, we now choose the irreducible local extensions $B_L, B_R \supset \mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_0$ to be maximal.


Let $\mathcal{B} \supset \mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_0$ local given by an algebra $\Theta_2$ and $\mathcal{A}_0$ completely rational and $Z$ be the coupling matrix $\theta_2 = \bigoplus Z_{\rho \sigma} \rho \boxtimes \bar{\sigma}$.

Then are equivalent:

▸ $\mathcal{B} \supset \mathcal{A}$ is maximal.

▸ $\mathcal{B} \supset \mathcal{A}$ is “modular invariant”, i.e. and $[Z, S] = [Z, T] = 0$.

▸ $\Theta_2 = Z(\Theta)$ is the “full center” $\cong \alpha$-induction construction of a (non-local) extension $\mathcal{B}_0 \supset \mathcal{A}_0$ on $\mathbb{R}$, i.e. an algebra $\Theta$ in $\text{Rep}(\mathcal{A}_0)$.

The chiral extension $\mathcal{B}_0 \supset \mathcal{A}_0$ is unique up to “Morita equivalence”.
Characterization maximal local extensions

We assume: $\mathcal{A}_+ = \mathcal{A}_- := \mathcal{A}_0$ completely rational.

Further, we now choose the irreducible local extensions $\mathcal{B}_L, \mathcal{B}_R \supset \mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_0$

to be maximal.

**Theorem** ((Rehren, Müger, Kawahigashi–Longo, Kong–Runkel, Carpi–K–L, B–K–L))

Let $\mathcal{B} \supset \mathcal{A} \equiv \mathcal{A}_0 \otimes \mathcal{A}_0$ local given by an algebra $\Theta_2$ and $\mathcal{A}_0$ completely rational and $Z$ be the coupling matrix $\theta_2 = \bigoplus Z_{\rho\sigma} \rho \boxtimes \bar{\sigma}$.

Then are equivalent:

- $\mathcal{B} \supset \mathcal{A}$ is maximal.
- $\mathcal{B} \supset \mathcal{A}$ is “modular invariant”, i.e. and $[Z, S] = [Z, T] = 0$.
- $\Theta_2 = Z(\Theta)$ is the “full center” $\cong$ “$\alpha$-induction construction” of a (non-local) extension $\mathcal{B}_0 \supset \mathcal{A}_0$ on $\mathbb{R}$, i.e. an algebra $\Theta$ in $\text{Rep}(\mathcal{A}_0)$.

The chiral extension $\mathcal{B}_0 \supset \mathcal{A}_0$ is unique up to “Morita equivalence”.
Classification of irreducible boundaries by chiral data

Given $B_L, B_R \supset A \equiv A_0 \otimes A_0$ maximal. Full center of
- $\Theta_L, \Theta_R$ algebras (Q-systems) in $\text{Rep}^I(A_0)$.
- equivalently (non-local) extension $B_{0,L}, B_{0,R} \supset A_0$ on $\mathbb{R}$.

Theorem

Universal $A$–topological $B_L$–$B_R$-boundary decomposes:

$$D_{\text{univ}} \cong \bigoplus_{[\beta]} D_{[\beta]}, \quad D_{[\beta]} \text{ irreducible}.$$

Sum indexed by (equivalently)
- isoclasses of irreducible $\Theta_L$–$\Theta_R$-bimodules $m_{\beta}$.
- irreducible sectors $[\beta] \in M_L C_{M_R}$, where $N = A_0(I)$, $M_L = B_{0,L}(I)$, $M_R = B_{0,R}(I)$, and $N C_N = \text{Rep}^I(A_0)$.

Classification by chiral data.
Classification of irreducible boundaries by chiral data

Given $\mathcal{B}_L, \mathcal{B}_R \supset \mathcal{A} \equiv \mathcal{A}_0 \otimes \mathcal{A}_0$ maximal. Full center of
- $\Theta_L, \Theta_R$ algebras (Q-systems) in $\text{Rep}^I(\mathcal{A}_0)$.
- equivalently (non-local) extension $\mathcal{B}_{0,L}, \mathcal{B}_{0,R} \supset \mathcal{A}_0$ on $\mathbb{R}$.

Theorem

**Universal $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{B}_R$-boundary decomposes:**

$$D_{\text{univ}} \cong \bigoplus_{[\beta]} D[\beta] , \quad D[\beta] \text{ irreducible}.$$  

**Sum indexed by (equivalently)**
- isoclasses of irreducible $\Theta_L$–$\Theta_R$-bimodules $m_\beta$.
- irreducible sectors $[\beta] \in M_L C_{M_R}$, where $N = \mathcal{A}_0(I), M_L = \mathcal{B}_{0,L}(I), M_R = \mathcal{B}_{0,R}(I)$, and $N C_N = \text{Rep}^I(\mathcal{A}_0)$.

Classification by chiral data.
Classification of irreducible boundaries by chiral data

Given $\mathcal{B}_L, \mathcal{B}_R \supset \mathcal{A} \equiv \mathcal{A}_0 \otimes \mathcal{A}_0$ maximal. Full center of

- $\Theta_L, \Theta_R$ algebras (Q-systems) in $\text{Rep}^I(\mathcal{A}_0)$.
- equivalently (non-local) extension $\mathcal{B}_{0,L}, \mathcal{B}_{0,R} \supset \mathcal{A}_0$ on $\mathbb{R}$.

**Theorem**

Universal $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{B}_R$-boundary decomposes:

$$D_{\text{univ}} \cong \bigoplus \limits_{[\beta]} D[\beta], \quad D[\beta] \text{ irreducible.}$$

**Sum indexed by** (equivalently)

- isoclasses of irreducible $\Theta_L$–$\Theta_R$-bimodules $m_\beta$.
- irreducible sectors $[\beta] \in M_L C_{M_R}$, where $N = A_0(I)$, $M_L = B_{0,L}(I)$, $M_R = B_{0,R}(I)$, and $N C_N = \text{Rep}^I(\mathcal{A}_0)$.

**Classification by chiral data.**
Classification of irreducible boundaries by chiral data

Given $\mathcal{B}_L, \mathcal{B}_R \supset \mathcal{A} \equiv \mathcal{A}_0 \otimes \mathcal{A}_0$ maximal. Full center of

- $\Theta_L, \Theta_R$ algebras (Q-systems) in $\text{Rep}^I(\mathcal{A}_0)$.
- equivalently (non-local) extension $\mathcal{B}_{0,L}, \mathcal{B}_{0,R} \supset \mathcal{A}_0$ on $\mathbb{R}$.

Theorem

Universal $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{B}_R$-boundary decomposes:

$$\mathcal{D}_{\text{univ}} \cong \bigoplus_{[\beta]} \mathcal{D}[\beta], \quad \mathcal{D}[\beta] \text{ irreducible.}$$

Sum indexed by (equivalently)

- isoclasses of irreducible $\Theta_L$–$\Theta_R$-bimodules $m_\beta$.
- irreducible sectors $[\beta] \in M_L C M_R$, where $N = \mathcal{A}_0(I)$, $M_L = \mathcal{B}_{0,L}(I)$, $M_R = \mathcal{B}_{0,R}(I)$, and $N C N = \text{Rep}^I(\mathcal{A}_0)$.

Classification by chiral data.
Let $\Theta_i = (\theta_i, w_i, x_i)$ with $i = 1, 2$ be two algebras/Q-systems in $N\mathcal{C}_N$. We define the **braided product** algebras/Q-systems $\Theta_1 \circ^\pm \Theta_2 = (\theta_1 \circ \theta_2, w_1 w_2, x_\pm)$, where

\[
\begin{array}{c}
\theta_1 \theta_2 & \theta_1 \theta_2 \\
\uparrow & \uparrow \\
x_+ & x_+ \\
\theta_1 \theta_2 & \theta_1 \theta_2 \\
\end{array} = \begin{array}{c}
\theta_1 \theta_2 & \theta_1 \theta_2 \\
\uparrow & \uparrow \\
x_1 & x_2 \\
\theta_1 \theta_2 & \theta_1 \theta_2 \\
\end{array}.
\]

**Definition**

**Universal $\mathcal{A}$-topological $\mathcal{B}_L$–$\mathcal{B}_R$ phase boundary $D_{\text{univ}}$:**

$\Theta_{\text{univ}} = \Theta_L \circ^+ \Theta_R \leftrightarrow D_{\text{univ}} \supset \mathcal{A}$, \hspace{1cm} $\Theta_L \leftrightarrow \mathcal{B}_L \supset \mathcal{A}$, \hspace{1cm} $\Theta_R \leftrightarrow \mathcal{B}_R \supset \mathcal{A}$
Let $\Theta_i = (\theta_i, w_i, x_i)$ with $i = 1, 2$ be two algebras/Q-systems in $\mathcal{NC}_N$. We define the **braided product** algebras/Q-systems $\Theta_1 \circ^{\pm} \Theta_2 = (\theta_1 \circ \theta_2, w_1 w_2, x_{\pm})$, where

$$
\begin{align*}
\theta_1 \theta_2 & \quad \theta_1 \theta_2 \\
x_+ & = \\
\theta_1 \theta_2 & \quad \theta_1 \theta_2 \\
x_1 & \\x_2
\end{align*}
$$

**Definition**

Universal $\mathcal{A}$-topological $\mathcal{B}_L$-$\mathcal{B}_R$ phase boundary $\mathcal{D}_{\text{univ}}$:

$$\Theta_{\text{univ}} = \Theta_L \circ^{+} \Theta_R \leftrightarrow \mathcal{D}_{\text{univ}} \supset \mathcal{A}, \quad \Theta_L \leftrightarrow \mathcal{B}_L \supset \mathcal{A}, \quad \Theta_R \leftrightarrow \mathcal{B}_R \supset \mathcal{A}$$
The full center construction

Let $\Theta_{LR}$ be the canonical algebra in $\mathcal{N} \mathcal{C}_N \boxtimes \overline{\mathcal{N} \mathcal{C}_N}$ with

$$
\Theta_{LR} = \bigoplus_{\rho} \rho \boxtimes \bar{\rho}.
$$

For $\Theta$ an algebra in $\mathcal{N} \mathcal{C}_N$ define the algebra $R(\Theta)$ in $\mathcal{N} \mathcal{C}_N \boxtimes \overline{\mathcal{N} \mathcal{C}_N}$

$$
R(\Theta) = (\Theta \boxtimes \text{id}_N) \circ^+ \Theta_{LR}
$$

$P^+_{R(\Theta)}$ projection $\text{Hom}(R(\Theta), R(\Theta))$:

$$
P^+_{R(\Theta)} = \begin{array}{c}
\begin{array}{c}
R(\theta)
\end{array}
\end{array},
$$

$$
\begin{array}{c}
\begin{array}{c}
R(\theta)
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
R(\theta) R(\theta) R(\theta)
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
R(\theta) R(\theta) R(\theta)
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
R(\theta) R(\theta) R(\theta)
\end{array}
\end{array}
$$

gives commutative subalgebra $\mathcal{Z}(\Theta)$, the full center.
The full center construction

Let $\Theta_{LR}$ be the canonical algebra in $\mathcal{N}C_N \boxtimes \overline{\mathcal{N}C_N}$ with

$$\Theta_{LR} = \bigoplus_{\rho} \rho \boxtimes \bar{\rho}.$$ 

For $\Theta$ an algebra in $\mathcal{N}C_N$ define the algebra $R(\Theta)$ in $\mathcal{N}C_N \boxtimes \overline{\mathcal{N}C_N}$

$$R(\Theta) = (\Theta \boxtimes \text{id}_N) \circ \Theta_{LR}.$$ 

$P^+_{R(\Theta)}$ projection $\text{Hom}(R(\Theta), R(\Theta))$:

$$P^+_{R(\Theta)} = \begin{cases} R(\theta) & R(\theta)R(\theta)R(\theta) & R(\theta)R(\theta)R(\theta) & R(\theta)R(\theta)R(\theta) \end{cases}$$

gives commutative subalgebra $Z(\Theta)$, the full center.
The full center construction

Let $\Theta_{LR}$ be the canonical algebra in $\mathcal{NC}_N \boxtimes \overline{\mathcal{NC}_N}$ with

$$\Theta_{LR} = \bigoplus_{\rho} \rho \boxtimes \bar{\rho}.$$

For $\Theta$ an algebra in $\mathcal{NC}_N$ define the algebra $R(\Theta)$ in $\mathcal{NC}_N \boxtimes \overline{\mathcal{NC}_N}$

$$R(\Theta) = (\Theta \boxtimes \text{id}_N) \circ^+ \Theta_{LR}.$$

$P^+_{R(\Theta)}$ projection $\text{Hom}(R(\Theta), R(\Theta))$:

$$P^+_{R(\Theta)} = \begin{array}{c}
\begin{array}{c}
R(\theta) \downarrow \\
R(\theta)
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
R(\theta) \downarrow \\
R(\theta)
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
R(\theta) \downarrow \\
R(\theta)
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
R(\theta) \downarrow \\
R(\theta)
\end{array}
\end{array}.$$ 

This gives commutative subalgebra $\mathcal{Z}(\Theta)$, the full center.
Sketch of proof

Theorem

Universal $\mathcal{A}$–topological $\mathcal{B}_L$–$\mathcal{B}_R$-boundary decomposes:

\[ \mathcal{D}_{\text{univ}} \cong \bigoplus_{[\beta]} \mathcal{D}_{[\beta]}, \quad \mathcal{D}_{[\beta]} \text{ irreducible.} \]

Sum indexed by (equivalently)

- isoclasses of irreducible $\Theta_L$–$\Theta_R$-bimodules $m_{\beta}$.
- irreducible sectors $[\beta] \in M_L C M_R$, where $N = A_0(I)$, $M_L = B_{0,L}(I)$, $M_R = B_{0,R}(I)$, and $N C N = \text{Rep}_I(A_0)$.

Classification by chiral data.
Sketch of proof

$m$ an irreducible $\Theta_L-\Theta_R$-bimodule gives intertwiner $\Theta_L \to \Theta_R$ lifting to $R(\Theta_L) \to R(\Theta_R)$

\[
\begin{array}{c}
\Theta_R \\
\circ \\
\Theta_L \\
\end{array}
\quad m, \quad D(m) =
\begin{array}{c}
R(\Theta_R) \\
\circ \\
R(\Theta_L) \\
\end{array}
\]

Invariant under projects:

\[
P^+_R(\Theta_R) D(m) P^+_R(\Theta_L) = P^+_R(\Theta_R) D(m) = D(m) P^+_R(\Theta_L) = D(m)
\]

$\sim D(m)$ restricts to $Z(\Theta_L) \to Z(\Theta_R)$. 
$m$ an irreducible $\Theta_L-\Theta_R$-bimodule gives intertwiner $\Theta_L \to \Theta_R$ lifting to $R(\Theta_L) \to R(\Theta_R)$

\[
\begin{array}{c}
\Theta_R \\
\bigcirc \\
\Theta_L
\end{array}
\quad m
\quad D(m) =
\begin{array}{c}
R(\Theta_R) \\
\bigcirc \\
R(\Theta_L)
\end{array}
\]

Invariant under projects:

\[
P^+_{R(\Theta_R)} D(m) P^+_{R(\Theta_L)} = P^+_{R(\Theta_R)} D(m) = D(m) P^+_{R(\Theta_L)} = D(m)
\]

$\sim D(m)$ restricts to $Z(\Theta_L) \to Z(\Theta_R)$. 
Generalized $S$-matrix (= usual one for Cardy case $\Theta_L = \Theta_R = \text{id}_N$)

\[ S_{e,m} \sim \begin{pmatrix} e^* \end{pmatrix} R(m) \]

\[
\begin{pmatrix}
S_{\rho\sigma} \sim \bar{\rho} \bigcirc \bigcirc \bar{\sigma} ;
\end{pmatrix}
\]

- orthonormal basis $e \in \text{Hom}(Z(\Theta_L), Z(\Theta_R))$.
- $m$ isoclasses of $\Theta_L$–$\Theta_R$-bimodules

**Theorem**

The matrix $(S_{e,m})$ is unitary.

In particular, the number irreducible of $\Theta_L$–$\Theta_R$-bimodules equals

\[
\dim \text{Hom}(Z(\Theta_L), Z(\Theta_R)) = \text{tr}(Z_L Z_R^t).
\]

Here, $Z_\bullet = \text{modular invariant matrix} Z(\Theta_\bullet) = \bigoplus Z_{\mu\nu\mu} \otimes \bar{\nu}$. 
Generalized S-matrix (= usual one for Cardy case $\Theta_L = \Theta_R = \text{id}_N$)

$$S_{e,m} \sim \begin{array}{ccc} e^* & \circlearrowleft & \circlearrowright \end{array} R(m)$$

$$\begin{pmatrix} S_{\rho\sigma} \sim \tilde{\rho} \bigcirc \bigcirc \tilde{\sigma} \end{pmatrix}$$

- orthonormal basis $e \in \text{Hom}(Z(\Theta_L), Z(\Theta_R))$.
- $m$ isoclasses of $\Theta_L$–$\Theta_R$-bimodules

Theorem

The matrix $(S_{e,m})$ is unitary.

In particular, the number irreducible of $\Theta_L$–$\Theta_R$-bimodules equals

$$\dim \text{Hom}(Z(\Theta_L), Z(\Theta_R)) = \text{tr}(Z_L Z_R^t).$$

Here, $Z_\bullet = \text{modular invariant matrix}$ $Z(\Theta_\bullet) = \bigoplus Z_{\bullet, \mu\nu} \mu \boxtimes \nu$. 
The decomposition of $\Theta_{\text{univ}} \equiv Z(\Theta_L) \circ^+ Z(\Theta_R)$ (killing ring trick):

\[
\sum_m \bar{S}_{p,m} = \sum_{m,e} \bar{S}_{p,m} S_{e,m} \delta_{e,p}
\]

(here $p$ unique projection $p : Z(\Theta_L) \to \text{id}_N \boxtimes \text{id}_N \to Z(\Theta_R)$) is the central decomposition of the algebra $\Theta_{\text{univ}}$, i.e. each summand is a projector associated to a subalgebra $\Theta_m$ of $\Theta_{\text{univ}}$ corresponding to $\mathcal{D}_m \supset \mathcal{A}$ where $\mathcal{D}_{\text{univ}} = \bigoplus_m \mathcal{D}_m \supset \mathcal{A}$. 
Fusion of defects

Fusion product by a braided relative product of algebras is well-defined, i.e. preserves left-right locality.

**Defects:** $D, \mathcal{E} = \mathcal{A}$-topological $\mathcal{B}$ defects give a new $\mathcal{A}$-topological $\mathcal{B}$ defects

$$D \boxtimes_\mathcal{B} \mathcal{E}.$$ 

**Phase boundaries:**

- $D = \mathcal{A}$-topological $\mathcal{B}_L$-$\mathcal{B}$ phase boundary
- $\mathcal{E} = \mathcal{A}$-topological $\mathcal{B}$-$\mathcal{B}_R$ phase boundary

give a $\mathcal{A}$-topological $\mathcal{B}_L$–$\mathcal{B}_R$ phase boundary:

$$D \boxtimes_\mathcal{B} \mathcal{E}.$$ 

Using $D(n) \cdot D(m) = D(m \otimes_\Theta n)$ (relative tensor product) gives:

$$D_m \boxtimes D_n \cong D_{m \otimes_\Theta n} := \bigoplus_{k \prec m \otimes_\Theta n} D_m,$$

i.e. fusion $\Leftrightarrow$ relative tensor product of bimodules.
Fusion of defects

Fusion product by a braided relative product of algebras is well-defined, i.e. preserves left-right locality.

**Defects:** \( \mathcal{D}, \mathcal{E} = \mathcal{A}\text{-topological } \mathcal{B} \) defects give a new \( \mathcal{A}\text{-topological } \mathcal{B} \) defects

\[ \mathcal{D} \boxtimes_{\mathcal{B}} \mathcal{E}. \]

**Phase boundaries:**

- \( \mathcal{D} = \mathcal{A}\text{-topological } \mathcal{B}_{L}\text{-}\mathcal{B}_{R} \) phase boundary
- \( \mathcal{E} = \mathcal{A}\text{-topological } \mathcal{B}_{L}\text{-}\mathcal{B}_{R} \) phase boundary

give a \( \mathcal{A}\text{-topological } \mathcal{B}_{L}\text{-}\mathcal{B}_{R} \) phase boundary:

\[ \mathcal{D} \boxtimes_{\mathcal{B}} \mathcal{E}. \]

Using \( D(n) \cdot D(m) = D(m \otimes_{\Theta} n) \) (relative tensor product) gives:

\[ \mathcal{D}_{m} \boxtimes \mathcal{D}_{n} \cong \mathcal{D}_{m \otimes_{\Theta} n} := \bigoplus_{k \prec m \otimes_{\Theta} n} \mathcal{D}_{m}, \]

i.e. fusion \( \Leftrightarrow \) relative tensor product of bimodules.
Fusion of defects

Fusion product by a braided relative product of algebras is well-defined, i.e. preserves left-right locality.

**Defects:** $\mathcal{D}, \mathcal{E} = \mathcal{A}$-topological $\mathcal{B}$ defects give a new $\mathcal{A}$-topological $\mathcal{B}$ defects

$$\mathcal{D} \boxtimes_{\mathcal{B}} \mathcal{E}.$$ 

**Phase boundaries:**

- $\mathcal{D} = \mathcal{A}$-topological $\mathcal{B}_L$-$\mathcal{B}$ phase boundary
- $\mathcal{E} = \mathcal{A}$-topological $\mathcal{B}$-$\mathcal{B}_R$ phase boundary

give a $\mathcal{A}$-topological $\mathcal{B}_L$-$\mathcal{B}_R$ phase boundary:

$$\mathcal{D} \boxtimes_{\mathcal{B}} \mathcal{E}.$$ 

Using $\mathcal{D}(n) \cdot \mathcal{D}(m) = \mathcal{D}(m \otimes_{\Theta} n)$ (relative tensor product) gives:

$$\mathcal{D}_m \boxtimes \mathcal{D}_n \cong \mathcal{D}_{m \otimes_{\Theta} n} := \bigoplus_{k < m \otimes_{\Theta} n} \mathcal{D}_m,$$

i.e. fusion $\iff$ relative tensor product of bimodules.
Fusion of defects

Fusion product by a braided relative product of algebras is well-defined, i.e. preserves left-right locality.

**Defects:** $\mathcal{D}, \mathcal{E} = \mathcal{A}$-topological $\mathcal{B}$ defects give a new $\mathcal{A}$-topological $\mathcal{B}$ defects

$$\mathcal{D} \boxtimes_{\mathcal{B}} \mathcal{E}.$$ 

**Phase boundaries:**

- $\mathcal{D} = \mathcal{A}$-topological $\mathcal{B}_L - \mathcal{B}$ phase boundary
- $\mathcal{E} = \mathcal{A}$-topological $\mathcal{B} - \mathcal{B}_R$ phase boundary

give a $\mathcal{A}$-topological $\mathcal{B}_L - \mathcal{B}_R$ phase boundary:

$$\mathcal{D} \boxtimes_{\mathcal{B}} \mathcal{E}.$$ 

Using $D(n) \cdot D(m) = D(m \otimes_\Theta n)$ (relative tensor product) gives:

$$\mathcal{D}_m \boxtimes \mathcal{D}_n \cong \mathcal{D}_{m \otimes_\Theta n} := \bigoplus_{k < m \otimes_\Theta n} \mathcal{D}_m,$$

i.e. fusion $\Leftrightarrow$ relative tensor product of bimodules.
Verlinde formula in MTCs

\[ N \mathcal{C}_N \text{ MTC, } N \Delta_N = \{ \text{id}, \rho_1, \ldots, \rho_n \} \] irreducible sectors (skeleton).

Fusion coefficients \( N^\tau_{\rho\sigma} = 0, 1, 2, \ldots \):

\[
[\mu] \times [\nu] = [\mu \nu] = \bigoplus_{\tau \in N \Delta_N} N^\tau_{\mu \nu} [\tau]
\]

**Verlinde formula** says that the S-matrix **diagonalizes** the fusion matrix \( N_\mu = (N^\tau_{\mu \nu})_{\{\nu, \tau \in N \Delta_N\}} \):

\[
N_\mu = SD_\mu S^* \quad D_\mu = \text{diag} \left( \frac{S_{\mu, \text{id}}}{S_{\text{id}, \text{id}}}, \frac{S_{\mu, \rho_1}}{S_{\text{id}, \rho_1}}, \ldots, \frac{S_{\mu, \rho_n}}{S_{\text{id}, \rho_n}} \right)
\]

Complex Fusion rule algebra commutative:

\[
\text{FuRu}(N \mathcal{C}_N) = \bigoplus_{i=0}^{n} \mathbb{C}
\]
Generalized Verlinde formula/Cardy formula for defects/phase boundaries: generalized S-matrices block-diagonalize the fusion rules.

Defects between the same phase: $\Theta$ in $\mathcal{N}\mathcal{C}_N \leftrightarrow N \subset M$. Fusion category $\Rightarrow$ dual category $\mathcal{M}\mathcal{C}_M$:

$$\text{FuRu}(\mathcal{M}\mathcal{C}_M) \cong \text{Hom}(Z(\Theta), Z(\Theta)) \cong \bigoplus_{\mu,\nu} \text{Mat}_{Z_{\mu\nu}}(\mathbb{C})$$

Boundaries between different phases $\Theta_a, \Theta_b, \leftrightarrow N \subset M_a, M_b$.

“Fusion algebroid”:

$$\text{FuRu}(\mathcal{M}_a\mathcal{C}_M\mathcal{b}) \cong \text{Hom}(Z(\Theta_a), Z(\Theta_b)) \cong \bigoplus_{\mu\nu} \text{Mat}_{Z_{a\mu\nu} \times Z_{b\mu\nu}}(\mathbb{C})$$
\[ A_0 \] completely rational, \( N = A_0(I), \) \( N C_N = \text{Rep}^I(\mathcal{A}), \) \( \mathcal{A} = A_0 \otimes A_0 \)

<table>
<thead>
<tr>
<th>braided subfactors</th>
<th>categorical</th>
<th>algebraic CQFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>((N \subset M, N C_N))</td>
<td>(\Theta) algebra in (N C_N)</td>
<td>full CFT (B \supset A)</td>
</tr>
<tr>
<td>(M C_M)</td>
<td>(\Theta-\Theta) bimodules</td>
<td>(A)-topological defect of (B)</td>
</tr>
<tr>
<td>(M_L C_M M_R)</td>
<td>(\Theta_L-\Theta_R) bimodules</td>
<td>(A)-top (B_L - B_R) boundary</td>
</tr>
</tbody>
</table>

composition: \(\beta \circ \gamma\), where \(\beta \in M_L C_M\), \(\gamma \in M_C M_R\)

relative \(\otimes\) product: \(m \otimes_\Theta n\), where \(m = \Theta_L-\Theta\) bimodule, \(n = \Theta-\Theta_R\) bimodule

intertwiner \(\beta \otimes \gamma\) \(D \boxtimes_B E\), where \(D = A\)-top \(B_L - B_R\) boundary, \(E = A\)-top \(B - B_R\) boundary

bimodule intertwiner boundary intertwiner
Virasoro net $\mathcal{A}_0 = \text{Vir}_{\frac{1}{2}}$:

- $\text{Vir}_{c=\frac{1}{2}}$ (= even part of the free real Fermion net).
- Sectors: $[\text{id}]$ (vacuum), $[\varepsilon]$ ($h = 1/2$), $[\sigma]$ ($h = 1/16$).
- Fusion rules: $\varepsilon \circ \varepsilon \cong \text{id}$, $\varepsilon \circ \sigma \cong \sigma$, $\sigma \circ \sigma \cong \text{id} \oplus \varepsilon$

Consider Ising model $\mathcal{B}_L = \mathcal{B}_R = \mathcal{B}_{LR}$, where

- $\mathcal{B}_{LR}(O) = \text{Vir}_{\frac{1}{2}}(I) \otimes \text{Vir}_{\frac{1}{2}}(J) \vee \{\Psi_\varepsilon, \Psi_\sigma\}$, $\mathcal{H}_{LR} = \bigoplus_{\rho=0,\varepsilon,\sigma} \mathcal{H}_\rho \otimes \mathcal{H}_\rho$
- Charged intertwinners (Cardy case bulk fields) of $\mathcal{B}_{LR}$:
  $\Psi_\varepsilon : \iota \rightarrow \iota \circ (\varepsilon \otimes \varepsilon)$, $\Psi_\sigma : \iota \rightarrow \iota \circ (\sigma \otimes \sigma)$.

Irreducible topological $\mathcal{B}_{LR} - \overline{\mathcal{B}_{LR}}$-boundaries/defects:

<table>
<thead>
<tr>
<th>Type</th>
<th>$\Psi^L$</th>
<th>$\Psi^R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>invisible</td>
<td>$\Psi_\varepsilon^L = \Psi_\varepsilon^R$</td>
<td>$\Psi_\sigma^L = \Psi_\sigma^R$</td>
</tr>
<tr>
<td>spin-flip</td>
<td>$\Psi_\varepsilon^L = \Psi_\varepsilon^R$</td>
<td>$\Psi_\sigma^L = -\Psi_\sigma^R$</td>
</tr>
<tr>
<td>order-disorder</td>
<td>$\Psi_\varepsilon^L = -\Psi_\varepsilon^R$</td>
<td>$\Psi_\sigma^L \neq \Psi_\sigma^R$</td>
</tr>
</tbody>
</table>
Categorical picture

- Same boundaries/defects as in the TFT construction of full CFTs on Riemann surfaces (Fuchs, Runkel, Schweigert (2002+)).

- (Conjecturally) related to the functoriality of the center construction (Davydov, Kong, Runkel (2013)).
Summary

- Boundaries: *locality* and invisibility for subnet $A$ (*conservation*).
- Existence of *universal boundary* for finite index case.
- Classification of irreducible boundaries by “*chiral data*” in the rational maximal case.

Open problems

- **Fusion** of phase boundaries and defects, relation to *Connes Fusion product* and (Bartels, Douglas, Henriques (2013)).
- Phase boundaries *without* assuming conformal symmetry.
- 3+1D QFT and defects / local gauge transformations.
Completely rational nets on $\mathbb{R}$

- **Split property.** For every relatively compact inclusion of intervals $\exists$ intermediate type I factor $M$

  \[ \mathcal{A}_0 \left( \begin{array}{c} \infty \\ \infty \end{array} \right) \subset M \subset \mathcal{A}_0 \left( \begin{array}{c} \infty \\ \infty \end{array} \right) \]

- **Strong Additivity** ($\cong$ Haag duality on $\mathbb{R}$). For touching intervals:

  \[ \mathcal{A}_0 \left( \begin{array}{c} \infty \\ \infty \end{array} \right) \lor \mathcal{A}_0 \left( \begin{array}{c} \infty \\ \infty \end{array} \right) = \mathcal{A}_0 \left( \begin{array}{c} \infty \\ \infty \end{array} \right) \]

- **Finite $\mu$-index:** finite Jones index for the inclusion

  \[ \mathcal{A}_0 \left( \begin{array}{c} \infty \\ \infty \end{array} \right) \lor \mathcal{A}_0 \left( \begin{array}{c} \infty \\ \infty \end{array} \right) \subset \left( \mathcal{A}_0 \left( \begin{array}{c} \infty \\ \infty \end{array} \right) \lor \mathcal{A}_0 \left( \begin{array}{c} \infty \\ \infty \end{array} \right) \right)' \]

  where the net is extended to $S^1$ by $\mathcal{A}_0 \left( \begin{array}{c} \infty \\ \infty \end{array} \right) \equiv \mathcal{A}_0 \left( \begin{array}{c} \infty \\ \infty \end{array} \right)'$. 

\[ \text{Marcel Bischoff (Uni G"ottingen)} \]

\[ \text{Defects and Boundaries in Algebraic Conformal QFT} \]

\[ \text{Vanderbilt May 8, 2014} \]


Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. 
TFT construction of RCFT correlators. I. Partition functions. 

Y. Kawahigashi and Roberto Longo. 
Classification of local conformal nets. Case $c < 1$. 

Y. Kawahigashi, Roberto Longo, and Michael Müger. 
Multi-Interval Subfactors and Modularity of Representations in Conformal Field Theory. 

Roberto Longo. 
A duality for Hopf algebras and for subfactors. I. 
Antony Wassermann.
Operator algebras and conformal field theory III. Fusion of positive energy representations of LSU(N) using bounded operators.

Feng Xu.
Jones-Wassermann subfactors for disconnected intervals.