Topological Defects and Phase Boundaries in Algebraic Conformal Quantum Field Theory*

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*work in progress with R. Longo, Y. Kawahigashi and K.-H. Rehren

- Algebraic quantum field theory: A family of algebras containing all local observables/operations associated with space-time regions.

- Conformal Quantum Field Theory (CQFT) in 1 and 2 dimension described by AQFT quite successful, e.g. partial classification results (e.g. $c < 1$) (Kawahigashi, Longo 2004).

- Topological Field Theory (TFT) construction of CFT on Riemann surfaces with boundaries/defects (Fuchs, Runkel, Schweigert (2002+)).

Subfactor $N \subset M$, $[M : N] < \infty$, finite depth.

II ${}_M K_N := {}_M L^2 M_N$, ${}_N H_M$ its dual.

III $\iota\colon M \leftarrow N$ inclusion map, $\bar{\iota}\colon N \leftarrow M$ its dual.

Categories ${}_N\mathcal{C}_N, {}_M\mathcal{C}_N, {}_N\mathcal{C}_M, {}_M\mathcal{C}_M$ generated by

II Submodules $L \prec \cdots \boxtimes K \boxtimes H \boxtimes K \boxtimes \cdots$,
e.g. ${}_M\mathcal{C}_N = \{L \prec K \boxtimes (H \boxtimes K)^{\boxtimes n}\} \subset \mathrm{Bim}(M, N)$.

III Subsectors $\rho \prec \cdots \circ \iota \circ \bar{\iota} \circ \iota \circ \bar{\iota} \circ \cdots$, e.g.
${}_M\mathcal{C}_N = \{\rho \prec \iota \circ (\bar{\iota} \circ \iota)^{\circ n}\} \subset \mathrm{Mor}(N, M)$.

Morita equivalence of **fusion categories:** ${}_N\mathcal{C}_N \approx_{\mathrm{Morita}} {}_M\mathcal{C}_M$.
"Morita context" = 2-category:

- 0-cells: $\{N, M\}$
- 1-cells: objects in ${}_N\mathcal{C}_N, {}_M\mathcal{C}_N, {}_N\mathcal{C}_M, {}_M\mathcal{C}_M$.
- 2-cells: intertwiners/bimodule maps

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Conversely starting with given **(unitary) fusion category** ${}_N\mathcal{C}_N$

II ${}_N\mathcal{C}_N \subset \mathrm{Bim}(N,N)$ (full and replete)

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consider subfactor (extension) $N \subset M$, such that:

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There are only finitely many such irreducible $N \subset M$ (up to equivalence).

If ${}_N\mathcal{C}_N$ has a **braiding**, i.e. is a **(unitary) ribbon fusion category**, we call the pair $(N \subset M, {}_N\mathcal{C}_N)$ a **braided subfactor**.

If further the **braiding** is non-degenerate, i.e. ${}_N\mathcal{C}_N$ is a **(unitary) modular tensor category (UMTC)**, we call the pair $(N \subset M, {}_N\mathcal{C}_N)$ a **non-degeneratly braided subfactor**.

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Coordinate free description of **"Ocneanu cells"** $=$ 2-category:

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What is the connection to algebraic conformal QFT? Briefly:

- $N =$ hyperfinite type III_1 factor prescribing a local algebra of observables $N = \mathcal{A}(I)$ of a fixed net $I \mapsto \mathcal{A}(I)$.
- ${}_N\mathcal{C}_N =$ UMTC of Doplicher–Haag–Roberts representations of $\mathcal{A}$ localized in I.
- $N \subset M$ (up to Morita equivalence) specifies a full CFT on Minkowski space (talk: Kawahigashi).
- ${}_N\mathcal{C}_M$ describes boundary conditions (talk: Kawahigashi).
- ${}_M\mathcal{C}_M$ fusion category of defects
- ${}_{M_a}\mathcal{C}_{M_b}$ boundaries between two full CFTs associated with $N \subset M_a, M_b$, respectively.

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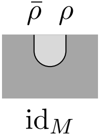
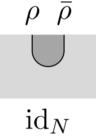
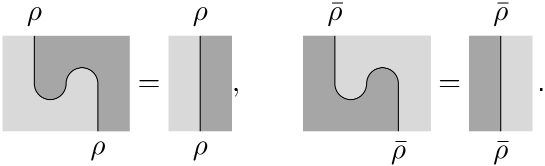
The 1-morphism $\bar{\rho} : N \to M$ is a **dual (conjugate)** of $\rho : M \to N$ if there exist intertwiners $R \in \mathrm{Hom}(\mathrm{id}_M, \bar{\rho}\rho)$ and $\bar{R} \in \mathrm{Hom}(\mathrm{id}_N, \rho\bar{\rho})$

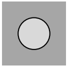
$R =$ 

$\bar{R} =$ 

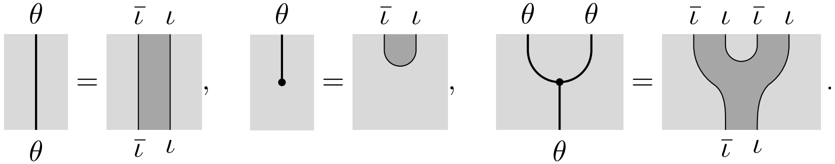
standard solution of the **zig-zag identities**:



Dimension of an 1-morphism = square root of index of subfactor:

$d\rho = d\bar{\rho} =$  $=$  $= [M : \rho(N)]^{\frac{1}{2}}$

Subfactor $N \subset M$, $[M : N] < \infty$, fin. depth. Define triple Θ in $\mathrm{End}(N)$



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Gives $\theta = \bar\iota \circ \iota$ structure of an **(co-)algebra** or **Q-system:**

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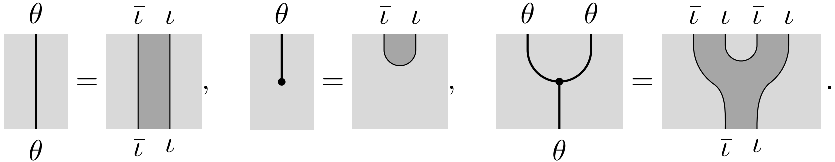

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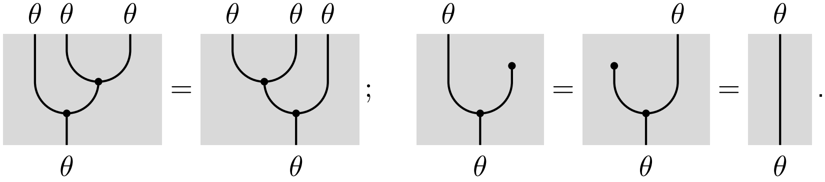

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Let N be a type III factor (Longo 1994) (or use bimodules)

$$\boxed{N \subset M} \longleftrightarrow \boxed{\text{algebra } \Theta \text{ in } \mathrm{End}(N)}$$

Namely, the Q-system defines a conditional expectation E on N and M is given by the Jones basic construction $E(N) \subset N \subset M$.

Fixing a URFC or UMTC ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$

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- ${}_N\mathcal{C}_M \cong$ category of (right) Θ-modules by tensoring on the right by ι.
- ${}_{M_a}\mathcal{C}_{M_b} \cong$ category of Θ_a-Θ_b bimodules by $\beta \longmapsto \bar{\iota}_a \circ \beta \circ \iota_b$.

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Preliminaries on AQFT

Boundaries / Defects

Classification (modular case)

1+1D (local) conformal net $\mathcal{A}$ on Minkowski space $\mathbb{R}^{1,1}$:

$$\mathbb{R}^{1,1} \supset O \mapsto \mathcal{A}(O) \subset \mathrm{B}(\mathcal{H}_{\mathcal{A}}), \qquad \mathcal{H}_{\mathcal{A}} \text{ fixed Hilbert space}$$

1. Isotony: $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
2. Locality: $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$ if O_1 and O_1 spacelike separated.
3. Covariance: $U_{\mathcal{A}}$ is a unitary **positive-energy** representation of the "2D conformal group", s.t. $U_{\mathcal{A}}(g)\mathcal{A}(O)U_{\mathcal{A}}(g)^* = \mathcal{A}(gO)$.
4. Vacuum: $\exists\ \Omega$ is unique translation invariant vector.

Chiral (local) conformal net $\mathcal{A}$ on $S^1 \cong \overline{\mathbb{R}}$ (compactified light-ray):

$$S_1 \supset I \mapsto \mathcal{A}(I) \subset \mathrm{B}(\mathcal{H}_{\mathcal{A}_0}), \qquad \mathcal{H}_{\mathcal{A}_0} \text{ fixed Hilbert space}$$

1. Isotony: $I \subset J \Rightarrow \mathcal{A}_0(I) \subset \mathcal{A}_0(J)$
2. Locality: $[\mathcal{A}_0(I), \mathcal{A}_0(J)] = \{0\}$ if $I \cap J = \emptyset$.
3. Covariance: $U_{\mathcal{A}_0}$ is a unitary **positive-energy** representation of the Möbius group/diffeomorphism group, s.t. $U_{\mathcal{A}_0}(g)\mathcal{A}_0(I)U_{\mathcal{A}_0}(g)^* = \mathcal{A}(gI)$.
4. Vacuum: $\exists\ \Omega$ is unique translation invariant vector.

Representation of $\mathcal{A} = \{\mathcal{A}(I)\}_{I\subset S^1}$ is a family:

$$\pi = \{\pi_I\colon \mathcal{A}(I) \to \mathrm{B}(\mathcal{H}_\pi)\}\,,$$

which is compatible, i.e. $\pi_J \restriction \mathcal{A}(I) = \pi_I$ for $I \subset J$.

- For all I_0 exists a $\rho \cong \pi$ on $\mathcal{H}$, s.t. $\rho_I = \mathrm{id}_{\mathcal{A}(I)}$ for $I \cap I_0 = \emptyset$.
- ρ_J is localized DHR endomorphism: $\rho_J(\mathcal{A}(J)) \subset \mathcal{A}(J)$ for all $J \supset I_0$.
- Tensor product: composition of localized endomorphisms.
- $\mathrm{Rep}^I(\mathcal{A}) \subset \mathrm{End}(\mathcal{A}(I))$ (full and replete).
- $\exists$ natural braiding (Fredenhagen, Rehren, Schroer (1989)).

Theorem ((Kawahigashi, Longo, Müger (2001)))

If $\mathcal{A}$ is a **completely rational** *conformal net.* ▸ Complete Rationality,
$\Longrightarrow \mathrm{Rep}(\mathcal{A})$ *is a* **modular C*-tensor category** / *UMTC.*

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Example

- Conformal nets with central charge $c < 1$.
- Conformal nets associated with even lattices (Dong, Xu 2006).
- Conformal nets associated with PER of loop groups ($\mathrm{SU}(N)$ (Wassermann 1998, Xu 2000), simply-laced groups level 1):

$$\mathcal{A}_{G,k}(I) = \pi(L_I G)''$$

- Finite index extensions (e.g. simple currents) and finite index subnets (e.g. finite orbifolds) of completely rational conformal nets.

Conjecture

Every unitary, C_2-cofinite VOA (+add. assumptions) gives a completely rational conformal net on S^1 and the associated MTCs of representations are equivalent.

Example

- Conformal nets with central charge $c < 1$.
- Conformal nets associated with even lattices (Dong, Xu 2006).
- Conformal nets associated with PER of loop groups ($\mathrm{SU}(N)$ (Wassermann 1998, Xu 2000), simply-laced groups level 1):

$$\mathcal{A}_{G,k}(I) = \pi(L_I G)''$$

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Every unitary, C_2-cofinite VOA (+add. assumptions) gives a completely rational conformal net on S^1 and the associated MTCs of representations are equivalent.

$\mathcal{B} \supset \mathcal{A}$ irreducible local extension with finite index $[\mathcal{B}(I) : \mathcal{A}(I)] < \infty$, then the algebra/Q-system Θ associated with $\mathcal{A}(I) \subset \mathcal{B}(I)$ is in $\mathrm{Rep}^I(\mathcal{A})$ and commutative, also the converse holds:

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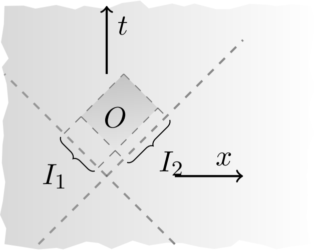

One can define a **1+1D conformal net** on **Minkowski space** by

$$\mathcal{A}(O) = \mathcal{A}_+(I_1)\otimes\mathcal{A}_-(I_2) \subset \mathrm{B}(\mathcal{H}_{\mathcal{A}_+}\otimes\mathcal{H}_{\mathcal{A}_-}),$$

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General 1+1D conformal nets are given by irreducible local extensions

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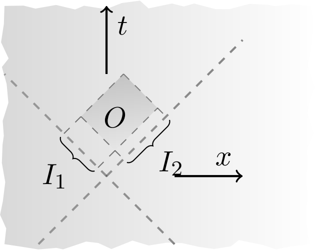

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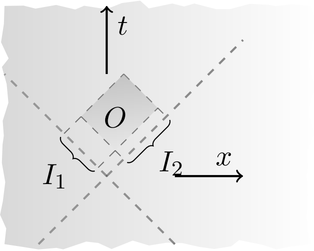

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Left observables $\mathcal{B}_{\mathrm{L}} \supset \mathcal{A}$. Right observables $\mathcal{B}_{\mathrm{R}} \supset \mathcal{A}$.

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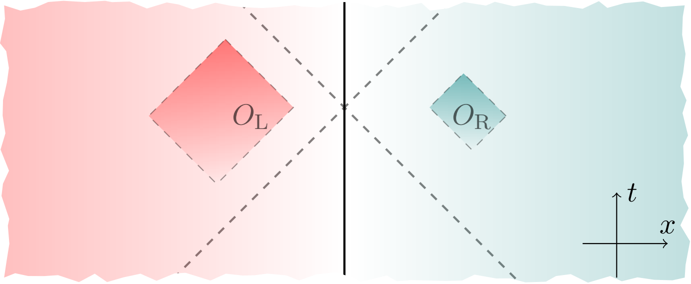

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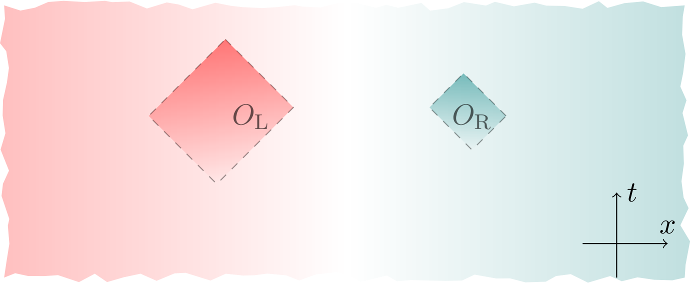

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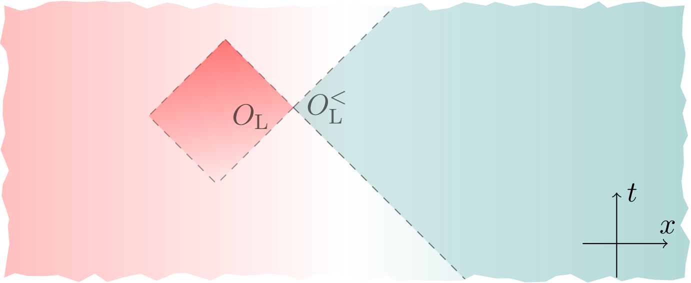

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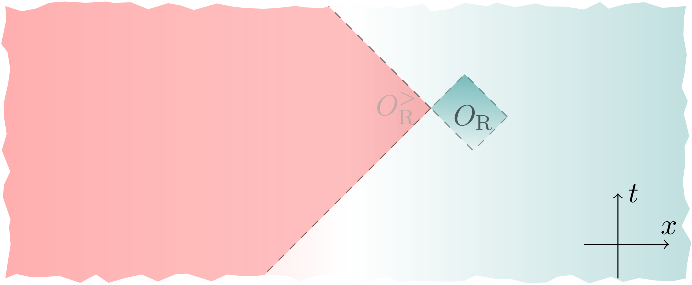

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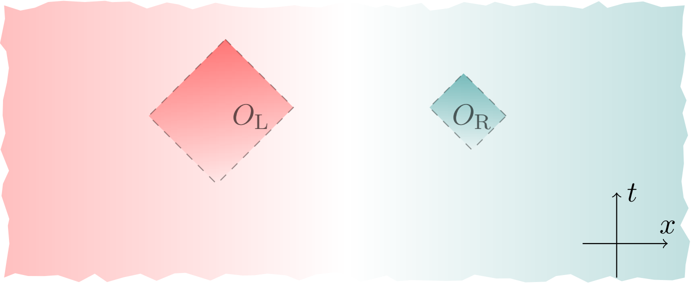

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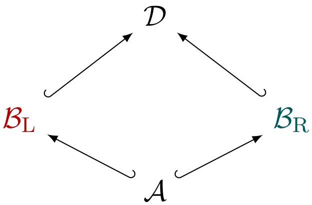

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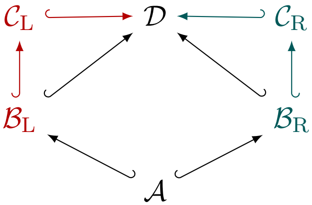

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Consider $\iota(N) \subset M$ with dual canonical endomorphism:

$$\theta = \bar{\iota} \circ \iota = \bigoplus_\rho n_\rho \cdot \rho\,.$$

- $M = N \vee \{\mathcal{H}_\rho\}_\rho \equiv N \vee \{\psi_{\rho,i}\}$,
- $\mathcal{H}_\rho := \mathrm{Hom}(\iota, \iota\rho)$ with ONB $\{\psi_{\rho,i}\}_{i=1}^{n_\rho}$.
- $\psi_{\rho,i}$ are charged intertwiners:

$$\psi_{\rho,i} \cdot \iota(n) = \iota(\rho(n)) \cdot \psi_{\rho,i} \qquad n \in N\,.$$

- Unique "Fourier decomposition":

$$m = \sum_{\rho,i} n_{\rho,i} \cdot \psi_{\rho,i} \qquad n_{\rho,i} \in N\,.$$

Universal $\mathcal{A}$–topological $\mathcal{B}_{\rm L}$–$\mathcal{B}_{\rm R}$ boundary $\mathcal{D}_{\rm univ}$, with no further relations between $\mathcal{B}_{\rm L}$ and $\mathcal{B}_{\rm R}$ besides locality:

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$$[\psi^{O_{\rm L}}_{{\rm L},i}, \psi^{O_{\rm R}}_{{\rm R},j}] = 0 \qquad \text{for all } i,j \text{ and } O_{\rm L} \text{ space-like left of } O_{\rm R}$$

Theorem

If $\mathcal{B}_{\rm L} \supset \mathcal{A}$ and $\mathcal{B}_{\rm R} \supset \mathcal{A}$ are local, irreducible, finite index extensions then there exists a unique $\mathcal{A}$–topological $\mathcal{B}_{\rm L}$–$\mathcal{B}_{\rm R}$ boundary $\mathcal{D}_{\rm univ}$ with the above properties.

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Morally: $\mathcal{D}_{\mathrm{univ}}$ is the **"fusion"** of

- $\mathcal{B}_{\mathrm{L}}$ as $\mathcal{A}$–topological $\mathcal{B}_{\mathrm{L}}$–$\mathcal{A}$ boundary and
- $\mathcal{B}_{\mathrm{R}}$ as $\mathcal{A}$–topological $\mathcal{A}$–$\mathcal{B}_{\mathrm{R}}$ boundary

over $\mathcal{A}$:

$${}_{\mathcal{B}_{\mathrm{L}}}[\mathcal{D}_{\mathrm{univ}}]_{\mathcal{B}_{\mathrm{R}}} = {}_{\mathcal{B}_{\mathrm{L}}}[\mathcal{B}_{\mathrm{L}}]_{\mathcal{A}} \underset{\mathcal{A}}{\boxtimes} {}_{\mathcal{A}}[\mathcal{B}_{\mathrm{R}}]_{\mathcal{B}_{\mathrm{R}}}.$$

Mathematically: $\mathcal{D}_{\mathrm{univ}}$ is obtained by a braided product of Q-systems specifying the inclusions $\mathcal{A} \subset \mathcal{B}_{\mathrm{L}}$ and $\mathcal{A} \subset \mathcal{B}_{\mathrm{R}}$, respectively.

Result

$\mathcal{D}_{\mathrm{univ}}(O)$ are general not factors, i.e. $\mathcal{D}_{\mathrm{univ}}$ is reducible. The center $\mathcal{D}_{\mathrm{univ}}(O) \cap \mathcal{D}_{\mathrm{univ}}(O)'$ is a finite algebra and the central decomposition of $\mathcal{D}_{\mathrm{univ}}$ gives "all" irreducible $\mathcal{A}$–topological $\mathcal{B}_{\mathrm{L}}$–$\mathcal{B}_{\mathrm{R}}$ boundaries/defects.

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Virasoro net $\mathcal{A}_0 = \mathrm{Vir}_{\frac{1}{2}}$:

- $\mathrm{Vir}_{c=\frac{1}{2}}$ (= even part of the free real Fermion net).
- Sectors: $[\mathrm{id}]$ (vacuum), $[\varepsilon]$ ($h = 1/2$), $[\sigma]$ ($h = 1/16$).
- Fusion rules: $\varepsilon \circ \varepsilon \cong \mathrm{id}, \quad \varepsilon \circ \sigma \cong \sigma, \quad \sigma \circ \sigma \cong \mathrm{id} \oplus \varepsilon$

Consider **Ising model** $\mathcal{B}_{\mathrm{L}} = \mathcal{B}_{\mathrm{R}} = \mathcal{B}_{\mathrm{LR}}$, where

- $\mathcal{B}_{\mathrm{LR}}(O) = \mathrm{Vir}_{\frac{1}{2}}(I) \otimes \mathrm{Vir}_{\frac{1}{2}}(J) \vee \{\Psi_\varepsilon, \Psi_\sigma\}, \quad \mathcal{H}_{\mathrm{LR}} = \bigoplus_{\rho=0,\varepsilon,\sigma} \mathcal{H}_\rho \otimes \mathcal{H}_\rho$
- Charged intertwinners (Cardy case bulk fields) of $\mathcal{B}_{\mathrm{LR}}$: $\Psi_\varepsilon : \iota \to \iota \circ (\varepsilon \otimes \varepsilon)$, $\Psi_\sigma : \iota \to \iota \circ (\sigma \otimes \sigma)$.

Irreducible topological $\mathcal{B}_{\mathrm{LR}}$–$\mathcal{B}_{\mathrm{LR}}$-boundaries/defects:

1	invisible	$\Psi^{\mathrm{L}}_\varepsilon = \Psi^{\mathrm{R}}_\varepsilon$	$\Psi^{\mathrm{L}}_\sigma = \Psi^{\mathrm{R}}_\sigma$
ε	spin-flip	$\Psi^{\mathrm{L}}_\varepsilon = \Psi^{\mathrm{R}}_\varepsilon$	$\Psi^{\mathrm{L}}_\sigma = -\Psi^{\mathrm{R}}_\sigma$
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Virasoro net $\mathcal{A}_0 = \mathrm{Vir}_{\frac{1}{2}}$**:**

- $\mathrm{Vir}_{c=\frac{1}{2}}$ (= even part of the free real Fermion net).
- Sectors: $[\mathrm{id}]$ (vacuum), $[\varepsilon]$ $(h = 1/2)$, $[\sigma]$ $(h = 1/16)$.
- Fusion rules: $\varepsilon \circ \varepsilon \cong \mathrm{id}, \quad \varepsilon \circ \sigma \cong \sigma, \quad \sigma \circ \sigma \cong \mathrm{id} \oplus \varepsilon$

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- $\mathcal{B}_{\mathrm{LR}}(O) = \mathrm{Vir}_{\frac{1}{2}}(I) \otimes \mathrm{Vir}_{\frac{1}{2}}(J) \vee \{\Psi_\varepsilon, \Psi_\sigma\}, \quad \mathcal{H}_{\mathrm{LR}} = \bigoplus_{\rho=0,\varepsilon,\sigma} \mathcal{H}_\rho \otimes \mathcal{H}_\rho$
- Charged intertwinners (Cardy case bulk fields) of $\mathcal{B}_{\mathrm{LR}}$: $\Psi_\varepsilon : \iota \to \iota \circ (\varepsilon \otimes \varepsilon)$, $\Psi_\sigma : \iota \to \iota \circ (\sigma \otimes \sigma)$.

Irreducible topological $\mathcal{B}_{\mathrm{LR}}$–$\mathcal{B}_{\mathrm{LR}}$-boundaries/defects:

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Preliminaries on AQFT

Boundaries / Defects

Classification (modular case)

We assume: $\mathcal{A}_+ = \mathcal{A}_- := \mathcal{A}_0$ **completely rational**.
Further, we now choose the irreducible local extensions

$$\mathcal{B}_\mathrm{L}, \mathcal{B}_\mathrm{R} \supset \mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_0$$

to be **maximal**.

Theorem ((Rehren, Müger, Kawahigashi–Longo, Kong–Runkel, Carpi–K–L, B–K–L))

Let $\mathcal{B} \supset \mathcal{A} \equiv \mathcal{A}_0 \otimes \mathcal{A}_0$ local given by an algebra Θ_2 and $\mathcal{A}_0$ completely rational and Z be the coupling matrix $\theta_2 = \bigoplus Z_{\rho\sigma} \rho \boxtimes \bar{\sigma}$.
Then are equivalent:

- $\mathcal{B} \supset \mathcal{A}$ *is* **maximal**.
- $\mathcal{B} \supset \mathcal{A}$ *is* **"modular invariant"**, *i.e. and* $[Z, S] = [Z, T] = 0$.
- $\Theta_2 = Z(\Theta)$ *is the* **"full center"** $\cong$ **"α-induction construction"** *of a (non-local) extension $\mathcal{B}_0 \supset \mathcal{A}_0$ on $\mathbb{R}$, i.e. an algebra Θ in $\mathrm{Rep}(\mathcal{A}_0)$.*

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Given $\mathcal{B}_{\mathrm{L}}, \mathcal{B}_{\mathrm{R}} \supset \mathcal{A} \equiv \mathcal{A}_0 \otimes \mathcal{A}_0$ maximal. Full center of

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Theorem

Universal $\mathcal{A}$–topological $\mathcal{B}_{\mathrm{L}}$–$\mathcal{B}_{\mathrm{R}}$-boundary decomposes:

$$\mathcal{D}_{\mathrm{univ}} \cong \bigoplus_{[\beta]} \mathcal{D}_{[\beta]}, \qquad \mathcal{D}_{[\beta]} \text{ irreducible.}$$

Sum indexed by (equivalently)

- *isoclasses of irreducible Θ_{L}–Θ_{R}-bimodules m_β.*
- *irreducible sectors $[\beta] \in {}_{M_{\mathrm{L}}}\mathcal{C}_{M_{\mathrm{R}}}$, where $N = \mathcal{A}_0(I)$, $M_{\mathrm{L}} = \mathcal{B}_{0,\mathrm{L}}(I)$, $M_{\mathrm{R}} = \mathcal{B}_{0,\mathrm{R}}(I)$, and ${}_N\mathcal{C}_N = \mathrm{Rep}^I(\mathcal{A}_0)$.*

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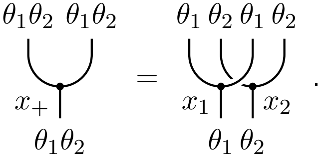
Let $\Theta_i = (\theta_i, w_i, x_i)$ with $i = 1, 2$ be two algebras/Q-systems in ${}_N\mathcal{C}_N$. We define the **braided product** algebras/Q-systems $\Theta_1 \circ^{\pm} \Theta_2 = (\theta_1 \circ \theta_2, w_1 w_2, x_{\pm})$, where



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Braided product of algebras/Q-systems

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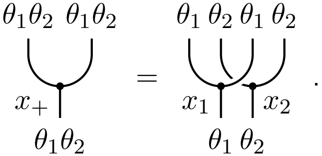

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$P^+_{R(\Theta)}$ projection $\mathrm{Hom}(R(\Theta), R(\Theta))$:

$P^+_{R(\Theta)} =$

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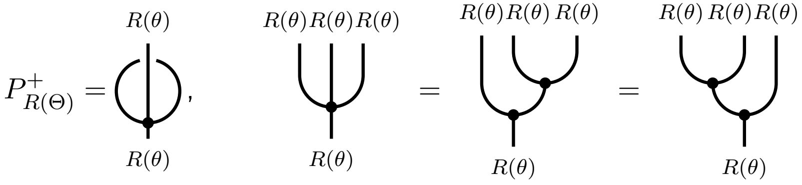

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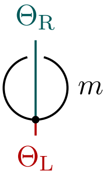
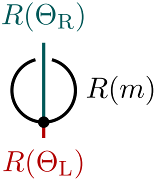
Sketch of proof

m an irreducible Θ_L–Θ_R-bimodule gives intertwiner $\Theta_\mathrm{L} \to \Theta_\mathrm{R}$ lifting to $R(\Theta_\mathrm{L}) \to R(\Theta_\mathrm{R})$



, $D(m) =$



Invariant under projects:

$$P^+_{R(\Theta_R)} D(m) P^+_{R(\Theta_L)} = P^+_{R(\Theta_R)} D(m) = D(m) P^+_{R(\Theta_L)} = D(m)$$

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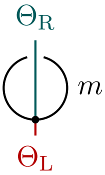

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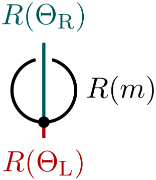

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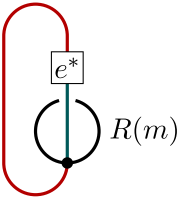
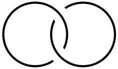
Generalized S-matrix (= usual one for Cardy case $\Theta_{\rm L} = \Theta_{\rm R} = \mathrm{id}_N$)

$S_{e,m} \sim$ 

$\Big(S_{\rho\sigma} \sim \ \bar\rho$  $\bar\sigma\ ; \Big)$

- orthonormal basis $e \in \mathrm{Hom}(Z(\Theta_{\rm L}), Z(\Theta_{\rm R}))$.
- m isoclasses of $\Theta_{\rm L}$–$\Theta_{\rm R}$-bimodules

Theorem

The matrix $(S_{e,m})$ is unitary.
In particular, the number irreducible of $\Theta_{\rm L}$–$\Theta_{\rm R}$-bimodules equals $\dim \mathrm{Hom}(Z(\Theta_{\rm L}), Z(\Theta_{\rm R})) = \mathrm{tr}(Z_{\rm L} Z_{\rm R}^t)$.

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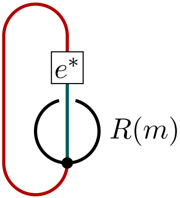
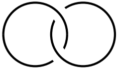
$S_{e,m} \sim$ 

$\Big(S_{\rho\sigma} \sim \ \bar\rho$  $\bar\sigma \ ; \Big)$

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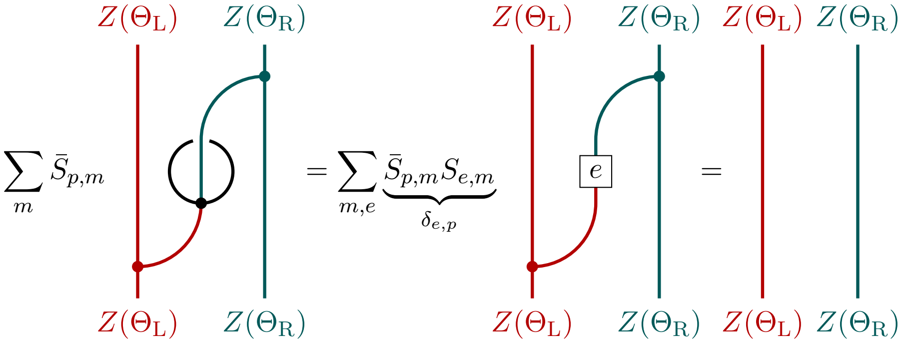
The decomposition of $\Theta_{\mathrm{univ}} \equiv Z(\Theta_{\mathrm{L}}) \circ^{+} Z(\Theta_{\mathrm{R}})$ (**killing ring** trick):



(here p unique projection $p\colon Z(\Theta_{\mathrm{L}}) \to \mathrm{id}_N \boxtimes \mathrm{id}_N \to Z(\Theta_{\mathrm{R}})$)
is the central decomposition of the algebra Θ_{univ}, i.e. each summand is a projector associated to a subalgebra Θ_m of Θ_{univ} corresponding to $\mathcal{D}_m \supset \mathcal{A}$ where $\mathcal{D}_{\mathrm{univ}} = \bigoplus_m \mathcal{D}_m \supset \mathcal{A}$.

Fusion product by a braided relative product of algebras is well-defined, i.e. preserves left-right locality.

Defects: $\mathcal{D}, \mathcal{E} = \mathcal{A}$-topological $\mathcal{B}$ defects give a new $\mathcal{A}$-topological $\mathcal{B}$ defects

$$\mathcal{D} \boxtimes_{\mathcal{B}} \mathcal{E}\,.$$

Phase boundaries:

- $\mathcal{D} = \mathcal{A}$-topological $\mathcal{B}_{\mathrm{L}}$-$\mathcal{B}$ phase boundary
- $\mathcal{E} = \mathcal{A}$-topological $\mathcal{B}$-$\mathcal{B}_{\mathrm{R}}$ phase boundary

give a $\mathcal{A}$-topologial $\mathcal{B}_{\mathrm{L}}$–$\mathcal{B}_{\mathrm{R}}$ phase boundary:

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Using $D(n) \cdot D(m) = D(m \otimes_{\Theta} n)$ (relative tensor product) gives:

$$\mathcal{D}_m \boxtimes \mathcal{D}_n \cong \mathcal{D}_{m \otimes_{\Theta} n} := \bigoplus_{k \prec m \otimes_{\Theta} n} \mathcal{D}_m\,,$$

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- $\mathcal{D} = \mathcal{A}$-topological $\mathcal{B}_{\mathrm{L}}$-$\mathcal{B}$ phase boundary
- $\mathcal{E} = \mathcal{A}$-topological $\mathcal{B}$-$\mathcal{B}_{\mathrm{R}}$ phase boundary

give a $\mathcal{A}$-topologial $\mathcal{B}_{\mathrm{L}}$–$\mathcal{B}_{\mathrm{R}}$ phase boundary:

$$\mathcal{D} \boxtimes_{\mathcal{B}} \mathcal{E} .$$

Using $D(n) \cdot D(m) = D(m \otimes_{\Theta} n)$ (relative tensor product) gives:

$$\mathcal{D}_m \boxtimes \mathcal{D}_n \cong \mathcal{D}_{m \otimes_\Theta n} := \bigoplus_{k \prec m \otimes_\Theta n} \mathcal{D}_m ,$$

i.e. fusion $\Leftrightarrow$ relative tensor product of bimodules.

${}_N\mathcal{C}_N$ MTC, ${}_N\Delta_N = \{\mathrm{id}, \rho_1, \ldots, \rho_n\}$ irreducible sectors (skeleton).
Fusion coefficients $N_{\rho\sigma}^{\tau} = 0, 1, 2, \ldots$:

$$[\mu] \times [\nu] = [\mu\nu] = \bigoplus_{\tau \in {}_N\Delta_N} N_{\mu\nu}^{\tau}[\tau]$$

Verlinde formula says that the S-matrix **diagonalizes** the fusion matrix $N_\mu = (N_{\mu\nu}^{\tau})_{\{\nu,\tau \in {}_N\Delta_N\}}$:

$$N_\mu = S D_\mu S^* \qquad D_\mu = \mathrm{diag}\left(\frac{S_{\mu,\mathrm{id}}}{S_{\mathrm{id},\mathrm{id}}}, \frac{S_{\mu,\rho_1}}{S_{\mathrm{id},\rho_1}}, \ldots, \frac{S_{\mu,\rho_n}}{S_{\mathrm{id},\rho_n}}\right)$$

Complex Fusion rule algebra commutative:

$$\mathrm{FuRu}({}_N\mathcal{C}_N) = \bigoplus_{i=0}^{n} \mathbb{C}$$

Generalized Verlinde formula/**Cardy formula** for defects/phase boundaries: generalized S-matrices **block-diagonalize** the fusion rules.

Defects between the same phase: Θ in ${}_N\mathcal{C}_N \leftrightarrow N \subset M$. Fusion category $=$ dual category ${}_M\mathcal{C}_M$:

$$\mathrm{FuRu}({}_M\mathcal{C}_M) \cong \mathrm{Hom}(Z(\Theta), Z(\Theta)) \cong \bigoplus_{\mu,\nu}^{n} \mathrm{Mat}_{Z_{\mu\nu}}(\mathbb{C})$$

Boundaries between different phases Θ_a, Θ_b, $\leftrightarrow N \subset M_a, M_b$.
"Fusion algebroid":

$$\mathrm{FuRu}({}_{M_a}\mathcal{C}_{M_b}) \cong \mathrm{Hom}(Z(\Theta_a), Z(\Theta_b)) \cong \bigoplus_{\mu\nu}^{n} \mathrm{Mat}_{Z^a_{\mu\nu}\times Z^b_{\mu,\nu}}(\mathbb{C})$$

$\mathcal{A}_0$ completely rational, $N = \mathcal{A}_0(I)$, ${}_N\mathcal{C}_N = \mathrm{Rep}^I(\mathcal{A})$, $\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_0$

braided subfactors	categorical	algebraic CQFT
$(N \subset M, {}_N\mathcal{C}_N)$	Θ algebra in ${}_N\mathcal{C}_N$	full CFT $\mathcal{B} \supset \mathcal{A}$
${}_M\mathcal{C}_M$	Θ-Θ bimodules	$\mathcal{A}$-topological defect of $\mathcal{B}$
${}_{M_\mathrm{L}}\mathcal{C}_{M_\mathrm{R}}$	Θ_L-Θ_R bimodules	$\mathcal{A}$-top $\mathcal{B}_\mathrm{L}$-$\mathcal{B}_\mathrm{R}$ boundary
composition: $\beta \circ \gamma$, where $\beta \in {}_{M_\mathrm{L}}\mathcal{C}_M$ $\gamma \in {}_M\mathcal{C}_{M_\mathrm{R}}$	relative $\otimes$ product: $m \otimes_\Theta n$, where $m = \Theta_\mathrm{L}$-Θ bimodule $n = \Theta$-Θ_R bimodule	fusion of boundaries: $\mathcal{D} \boxtimes_\mathcal{B} \mathcal{E}$, where $\mathcal{D} = \mathcal{A}$-top $\mathcal{B}_\mathrm{L}$-$\mathcal{B}$ boundary $\mathcal{E} = \mathcal{A}$-top $\mathcal{B}$-$\mathcal{B}_\mathrm{R}$ boundary
intertwiner	bimodule intertwiner	boundary intertwiner

Virasoro net $\mathcal{A}_0 = \mathrm{Vir}_{\frac{1}{2}}$:

- $\mathrm{Vir}_{c=\frac{1}{2}}$ (= even part of the free real Fermion net).
- Sectors: $[\mathrm{id}]$ (vacuum), $[\varepsilon]$ $(h = 1/2)$, $[\sigma]$ $(h = 1/16)$.
- Fusion rules: $\varepsilon \circ \varepsilon \cong \mathrm{id}, \quad \varepsilon \circ \sigma \cong \sigma, \quad \sigma \circ \sigma \cong \mathrm{id} \oplus \varepsilon$

Consider **Ising model** $\mathcal{B}_{\mathrm{L}} = \mathcal{B}_{\mathrm{R}} = \mathcal{B}_{\mathrm{LR}}$, where

- $\mathcal{B}_{\mathrm{LR}}(O) = \mathrm{Vir}_{\frac{1}{2}}(I) \otimes \mathrm{Vir}_{\frac{1}{2}}(J) \vee \{\Psi_\varepsilon, \Psi_\sigma\}, \quad \mathcal{H}_{\mathrm{LR}} = \bigoplus_{\rho=0,\varepsilon,\sigma} \mathcal{H}_\rho \otimes \mathcal{H}_\rho$
- Charged intertwinners (Cardy case bulk fields) of $\mathcal{B}_{\mathrm{LR}}$: $\Psi_\varepsilon : \iota \to \iota \circ (\varepsilon \otimes \varepsilon)$, $\Psi_\sigma : \iota \to \iota \circ (\sigma \otimes \sigma)$.

Irreducible topological $\mathcal{B}_{\mathrm{LR}}$–$\mathcal{B}_{\mathrm{LR}}$-boundaries/defects:

1	invisible	$\Psi^{\mathrm{L}}_\varepsilon = \Psi^{\mathrm{R}}_\varepsilon$	$\Psi^{\mathrm{L}}_\sigma = \Psi^{\mathrm{R}}_\sigma$
ε	spin-flip	$\Psi^{\mathrm{L}}_\varepsilon = \Psi^{\mathrm{R}}_\varepsilon$	$\Psi^{\mathrm{L}}_\sigma = -\Psi^{\mathrm{R}}_\sigma$
σ	order-disorder	$\Psi^{\mathrm{L}}_\varepsilon = -\Psi^{\mathrm{R}}_\varepsilon$	$\Psi^{\mathrm{L}}_\sigma \neq \Psi^{\mathrm{R}}_\sigma$

Categorical picture

- Same boundaries/defects as in the TFT construction of full CFTs on Riemann surfaces (Fuchs, Runkel, Schweigert (2002+)).

- (Conjecturally) related to the functoriality of the center construction (Davydov, Kong, Runkel (2013)).

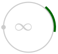
Summary

- Boundaries: **locality** and invisibility for subnet $\mathcal{A}$ (**conservation**).
- Existence of **universal boundary** for finite index case.
- Classification of irreducible boundaries by "**chiral data**" in the rational maximal case.

Open problems

- **Fusion** of phase boundaries and defects, relation to **Connes Fusion** product and (Bartels, Douglas, Henriques (2013)).
- Phase boundaries **without** assuming conformal symmetry.
- 3+1D QFT and defects / local gauge transformations.

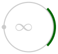
- Split property. For every relatively compact inclusion of intervals $\exists$ intermediate **type** I **factor** M

$\mathcal{A}_0\big($  $\big) \subset M \subset \mathcal{A}_0\big($  $\big)$

- Strong Additivity ($\cong$ Haag duality on $\mathbb{R}$). For touching intervals:

$\mathcal{A}_0\big($  $\big) \vee \mathcal{A}_0\big($  $\big) = \mathcal{A}_0\big($  $\big)$

- Finite μ-index: finite Jones index for the inclusion

$\mathcal{A}_0\big($  $\big) \vee \mathcal{A}_0\big($  $\big) \subset \Big(\mathcal{A}_0\big($  $\big) \vee \mathcal{A}_0\big($  $\big)\Big)'$

where the net is extended to S^1 by $\mathcal{A}_0\big($  $\big) \equiv \mathcal{A}_0\big($  $\big)'$.

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