Algebraic Quantum Field Theory

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Quantum Field Theory (QFT) is the general framework for the description of the physics of relativistic quantum systems, notably of elementary particles. It is the synthesis of Quantum Theory and Special Relativity, supplemented by the principle of Locality in space and time, and by the Spectral Condition in energy and momentum (see below). Algebraic QFT (AQFT) emphasizes the role of algebraic relations among observables which determine, rather than quantum fields proper, a physical system. Its foundations were laid in the 1960^s by R. Haag and D. Kastler, and the theory was develloped further by, among others, H.-J. Borchers and H. Araki.

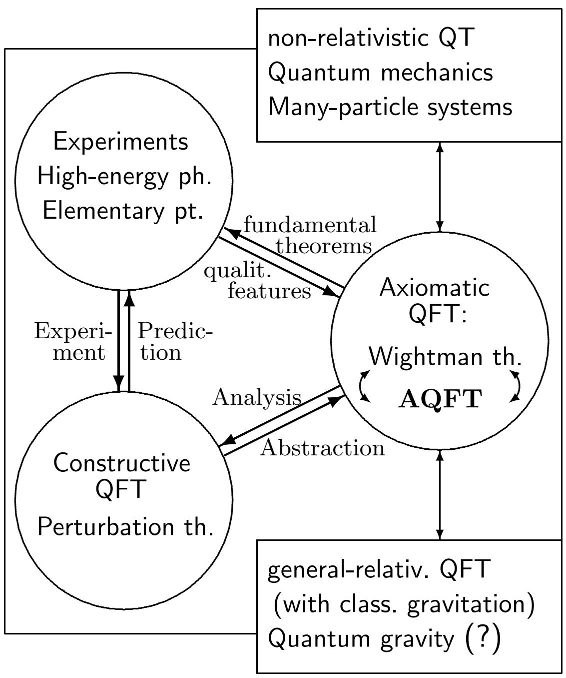

Fig. 1: *The position of AQFT within the theory of (special-)relativistic quantum fields*

Like the earlier ↗*Wightman theory*, AQFT analyzes the general features of quantum field theoretical models in the light of fundamental physical principles, in order to resolve conceptional difficulties with perturbative and euclidean (imaginary time) approaches. It formulates a general relativistic quantum theory and the relevant physical concepts and their interrelations in a model independent language. Its method is characterized by the description of phenomenological properties of a system in terms of comprehensive mathematical notions; this enables the application of mathematical theorems which acquire a physical significance. By its universality, the algebraic language qualifies best for the capture of universal principles of physics.

AQFT establishes analytic strategies in order to deduce structural properties of the systems under consideration, such as the particle content of the theory, from the algebra of observables and its representations. Particles are therefore considered as a dynamical feature rather than a kinematical datum coupled to the a priori fields of a phenomenological Lagrangean. In fact, AQFT does not use a Lagrangean, nor does it appeal to an underlying classical theory.

AQFT also enables the exact construction of simple relativistic models ($\mathrm{P}(\phi)_2$ and Yukawa$_2$ models as well as conformally invariant models in two dimensions). Its methods are likewise successfully applied to the study of phase transitions in non-relativistic models in Statistical Mechanics (e.g., spin systems).

The emphasis on the algebraic structure rather than on the fields of a theory responds to the fact that (in the language of Wightman theory) many different fields can be chosen which all describe the same scattering processes (Borchers classes); e.g., the odd Wick polynomials of a free massive field belong to the same class. As each field from a Borchers class generates the same local algebras, and as, vice versa, scattering states can be constructed solely in terms of local algebras, the latter have to be considered as the relevant entities. From this point of view, Wightman fields are – inspite of their ambiguity – a powerful tool in order to implement Locality and Poincaré Covariance (see below). The transition between the Wightman and the algebraic formulation is not yet completely controlled, but it is known that AQFT admits quantum systems which do not support actual fields. (For this reason, the name "Local Quantum Physics" is often preferred.)

Another motivation for the algebraic point of view arises from the fact that quantum systems with infinitely many degrees of freedom, unlike ↗*Quantum Mechanics*, possess many inequivalent physical representations. These describe classes of states between which there is no interference possible (superselection rules or sectors). The conceptional dichotomy between the algebra (specifying the physical system) and its representations (the possible preparations of the system) is a basic feature of AQFT and constitutes one of its conceptional advantages.

The axioms of AQFT: interpretation and consequences

AQFT formulates a system of *axioms*, among which we shall here distinguish three types:
– structural axioms, paying tribute to the quantum nature as well as to the local nature of the objects to be described (Hilbert space, localizability, see below);
– relativistic axioms, in order to comply with the principles of ↗*Special Relativity* (Causality, Covariance, Spectral Condition, see below), and
– a range of specific assumptions, which can be varied in order to distinguish and identify different types of theories or states (mass spectrum, Haag duality, phase space properties, inner symmetries, see below).

The axioms are not considered as absolute, but are instead themselves issues of research and subject to discussion within an evolving scientific area; they are designed to capture universal phenomenological evidence in the form of model-independent principles.

Observables and fields

↗*Quantum Theory* describes physical measurements and operations in terms of "observables". These are representable as selfadjoint operators on a Hilbert space, in which density matrices describe pure and mixed states. This setting guarantees the probabilistic interpretation, according to which matrix elements of operators are transition amplitudes or expectation values. In each representation, the bounded operators form a concrete C* algebra. The entirety of its representations can be characterized by an abstract C* algebra which is studied by means of the highly develloped mathematical theory of operator algebras.

In modern physics, all elementary particles and their interactions are described in terms of "relativistic quantum fields": these are operators varying with space and time, promoting ↗*M. Faraday*'s classical notion of *fields* with a dynamical law into Quantum Theory, in compliance with ↗*A. Einstein*'s theory of ↗*Special Relativity*. While not all quantum fields correspond to physically realizable operations (non-observable fields), yet all observables are expressed by them. AQFT takes care of the field concept by the assertion that a physical system is completely described by the specification of its *local* observables: for this purpose, with every region in space-time is associated the algebra of those observables which can be measured in the region; all observables can be approximated by local ones. It is avoided to rely on non-observable fields which as such are not accessible to measurement.

The local algebras cannot be independently specified. First of all, they must increase with their regions of localization (isotony). Furthermore, the propagation speed of dynamical processes being limited by the ↗*speed of light* is expressed by the axiom of Primitive Causality; it requires that the algebra associated with a region of arbitrarily narrow temporal extension contains all observables in the causal future and past of this region.

The mutual influence of two physical observables is reflected in algebraic commutation relations between the corresponding operators. In particular, observables localized in *causally independent* regions of space-time must be represented by *commuting* operators (Locality). Any algebraic structure beyond the algebraic coding of Locality distinguishes the physical system under consideration.

Unlike quantum mechanical particles, quantum fields comprise infinitely many degrees of freedom. Such systems possess many inequivalent representations, that is, different realizations by Hilbert space operators; the corresponding equivalence classes of states are called (superselection) sectors. Superselection sectors are a *global* concept: they are given by, e.g., a "total charge of the universe" which cannot be measured with finite experimental effort.

It is therefore appropriate to consider the abstract system as an algebraic structure; its concrete preparation in a global state (evoking an idealization) is described by the choice of the representation. Not all representations, however, are of physical interest: therefore suitable selection criteria have to be imposed on the states (see below).

Symmetries

Throughout in Quantum Theory, symmetries play a prominent role. Outer symmetries are symmetries of space and time according to Special Relativity, while inner symmetries relate, say, particles of equal mass but different charge. Inner symmetries are a most successful principle for phenomenological theories.

One distinguishes the symmetry of a system (that is, an algebraic automorphism) from its realization in specific states. It is conceivable that a system possesses a symmetry which in a given representation cannot be implemented by a unitary operator ("spontaneous symmetry breakdown"). In general, the invariance group of Special Relativity (↗*Poincaré group*: translations and Lorentz transformations, that is, rotations, boosts, and reflections) with the exception of space and time reflections is postulated as space-time symmetries of the algebra. The full symmetry is realized in states "of particle type" which are considered for the study of scattering processes; characteristic for such states is the vanishing mean energy density and finite total charge. In contradistinction, Lorentz symmetry is broken in thermal equilibrium states (see below).

In a translationally covariant representation, the operators of energy and momentum are well defined, and their spectrum can be studied. In physically reasonable states of particle type, the spectrum should be bounded below (Spectral Condition) since only a finite amount of energy can be extracted from a state with finitely many particles. In fact, the Lorentz covariance of the algebra is sufficient to establish the Lorentz invariance of the "lower edge" of the spectrum, even if this symmetry is broken. A representation contains a vacuum state if this lower edge is a point; a mass hyperboloid instead corresponds to a massive particle. Other possibilities characterize, e.g., massless particles or "infraparticles".

A prominent example for the algebraic interplay between spectral and invariance properties of states is the Goldstone theorem, according to which a continuous inner symmetry cannot be broken unless the representation contains massless excitations. AQFT allows to extend this theorem beyond its conventional field theoretic formulation, and to point out its limitations as well.

Unlike states of particle type, thermal states of a system of infinite size allow for the extraction of an arbitrary amount of energy. Therefore, the spectral condition is not adequate for thermal states. The ↗*Gibbs* formulation of the canonical ensemble which is successful in quantum statistics, is not applicable to systems of infinite size due to the continuous energy spectrum. It is replaced by the Kubo-Martin-Schwinger (KMS) condition which distinguishes thermal equilibrium states in terms of an analyticity property of the dynamical correlation functions.

Local algebras in a relativistic QFT must be algebras of type III (in the von-Neumann classification scheme) which do not possess minimal projections. This implies that – in contradistinction from non-relativistic systems – every state is *locally* impure; any global state may well be approximated by local operations with arbitrary precision, yet it always remains locally indistinguishable from an infinity of other states due to quantum fluctuations in the causal complement.

The global symmetries of space and time are (under suitable conditions) determined by the local algebras along with the vacuum state. This fact is due to an application of the "modular theory" of abstract von-Neumann algebras of type III, which associates with every state an adapted dynamics with respect to which this state is a thermal equilibrium state. For certain local algebras in the vacuum state, this dynamics is a subgroup of Lorentz transformations (boosts). The modular theory provides an algebraic proof of the ↗*CPT Theorem* (invariance under simultaneous inversions of space, time, and charge) as well as a new explanation of the ↗*Unruh effect* (an accelerated observer in Minkowski space-time perceives a temperature in the vacuum) and, in the presence of gravitational horizons, of the ↗*Hawking radiation* of black holes. Even the spectral condition can be formulated as a modular property of inclusions of local algebras.

A variety of specific assumptions concerns the "size" of local algebras of observables. The assumption that the observables in small space-time subregions generate the algebra of a large region (additivity), excludes theories exhibiting "fundamental units of length". Thermodynamical properties are anticipated in terms of assumptions on the size of ↗*phase space* (compactness, nuclearity); closely related is the problem whether different states can be prepared in causally disjoint regions (split property). AQFT establishes a multitude of logical relations between phase space assumptions and, e.g., the shape of mass spectra or the quantitative violation of ↗*Bell's inequalities*.

Superselection sectors

Along with the general possibility of superselection sectors arise the questions what sectors are realized in a given theory, and what physical interpretation they can be given.

Of particular interest for the theoretical analysis of superselection sectors is a property called Haag duality. It states that the local algebras are maximal in the sense that any assignment of further operators to them would violate the axiom of Locality. According to the set of regions for which Haag duality holds, there are weaker and stronger versions. Weak versions of Haag duality can be proven in many models, while their violation is a signal from spontaneously broken symmetries. A stronger version can be disproved in models with global but not local ↗*gauge symmetry*. The possibility is of great interest that ↗*Gauß' law* in local gauge theories might restore even strong Haag duality, and that conversely this property might characterize local gauge theories.

The theory of superselection sectors is founded on the basis of weak Haag duality. It leads to a general concept of charge and explains the laws governing charge composition. It provides an intrinsic description of (particle) statistics as well as a field-independent proof of the ↗*Theorem of Spin and Statistics* which correlates the ↗*Bose-Einstein* and ↗*Fermi-Dirac statistics* with integer and half-integer spin, respectively. Its most eminent result, however, is the proof of a symmetry principle: all superselection sectors "of particle type" can be attributed to a global inner gauge symmetry group acting on localized "charged fields" and preserving the vacuum; the observables are identical with the gauge invariants, and the superselection charges coincide with the gauge charges. The algebra of charged operators shares all abstract properties of observables, except causality being replaced by causal anti-commutativity in the presence of fermionic sectors. Hence, it is itself accessible to the methods of AQFT.

The class of sectors treated in the basic form of this theory is characterized by a – somewhat ad hoc – selection criterion, which claims the transportability of charges to arbitrarily distant bounded space-time regions. It has been proven, however, that all massive states which behave asymptotically like vacuum states indeed admit a transportable localization of charge at least in arbitrarily narrow regions of the shape of conical space-like tubes. Such charge tubes are interpreted as the algebraic manifestation of "gluon strings" in non-abelian gauge theories.

Various generalizations of the theory of superselection sectors have been established. Integrable models in one space dimension exhibit soliton sectors which asymptotically connect two different vacuum states. The Fermi-Bose alternative of statistics is relaxed in low-dimensional theories; the study of possible statistics and their realization in conformally invariant QFT has reveiled unforeseen interrelations with areas of pure mathematics (theory of knots and theory of subfactors).

Perspective

AQFT is also concerned with the problematization of the concept of particles. This concept in its usual form does not capture properly, e.g., the electron in ↗*Quantum-Electro-Dynamics* which cannot be separated from its surrounding "photon cloud". ↗*Quarks* in ↗*Quantum-Chromo-Dynamics* do not occur as isolated charged particles, being permanently confined by color forces. AQFT has develloped extensions of the particle concept in order to incorporate these particles.

The problematic issue of particles is considered as characteristic for local gauge theories. It cannot be separated from the difficulty that such theories in all their modern formulations necessarily rely on non-observable objects which defy a probabilistic interpretation in the sense of quantum theory, and which therefore fall beyond the range of validity of most mathematical propositions exploited in AQFT. The characterization of gauge theories in terms of algebraic properties solely of their observables, as well as the subsequent treatment by the methods of AQFT is still lacking.

The notions of AQFT are – with appropriate modifications – applicable to QFT on curved space-time, that is, the semi-classical theory of ↗*General Relativity*. Most results, however, which depend on Lorentz invariance and the spectral condition cannot be maintained since a curved space-time will in general not possess global symmetries. Alternative approaches in which local properties substitute global space-time symmetries are a current topic of research.

A true theory of Quantum Gravity in the spirit of AQFT will have to abandon the absolute causal structure of space-time, which in Quantum Gravity must itself be subject to the ↗*Heisenberg uncertainty*.

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