Weak C* Hopf symmetry

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Abstract

Weak C* Hopf algebras can act as global symmetries in low-dimensional quantum field theories, when braid group statistics prevents ordinary group symmetries. Charged fields transform linearly in finite multiplets, and the observables are precisely the gauge invariants. Possibilities to construct field algebras with weak C* Hopf symmetry from a given theory of local observables are discussed.

1 Introduction

On several occasions at the Arnold Sommerfeld Institute at Clausthal (and elsewhere) I have addressed the field problem and the symmetry problem in the theory of superselection sectors of low-dimensional models of quantized fields [1,2].

The field problem raises the question whether the algebra of physical observables can be embedded into a field algebra containing sufficiently many charged fields to generate from the vacuum all superselection sectors of the observables. The identification of a sensible class of gauge symmetries in a given model with braid group statistics (which precludes symmetry groups) constitutes the symmetry problem.

The field problem can always be solved by a canonical construction of intertwining field operators, which was introduced in [3] and called the "reduced field bundle" (with commutation relations in the form of an exchange algebra). Yet, I have myself rejected this answer as a useful candidate for the ambient field algebra. The reason was the apparent absence of a decent symmetry under which the reduced field bundle transforms, with the observables as the invariants (the "gauge principle").

A main point in this report is the observation that the reduced field bundle does possess a symmetry which I have previously overlooked. In fact, it is invariant under the action of a weak C* Hopf algebra. Weak C* Hopf algebras [4,5] (and related objects [6]) are comparatively conservative generalizations of finite symmetry groups. The coproduct is non-commutative and does not map the unit operator onto the unit operator. Unlike quantum groups [7], however, it preserves the $*$-structure, and unlike quasi quantum groups [8], it is coassociative. Unlike the combinatorial concept of a paragroup [9], and unlike my personal attempts with C* symmetries [2], its action on the field algebra is determined by a linear transformation law on finite tensor multiplets of charged fields.

Weak C* Hopf symmetries are by no means restrained to the reduced field bundle. They are rather the natural symmetry concept associated with finite index subfactors of depth 2. If such a subfactor is irreducible, then the symmetry is in fact a C* Hopf algebra; so its "weakness" is precisely due to reducibility. The depth 2 condition has rather a technical meaning ("all irreducible components of the subfactor $A \subset B_2$ are already contained in $A \subset B$"), and in the case of irreducible subfactors becomes a simple structural property ("$A' \cap B_2$ is a factor"). Here B_2 is the second Jones extension of $A \subset B$.

Weak C* Hopf algebras possess unitary matrix representations. The size of the corresponding multiplets (of charged fields) is, in contrast to the true Hopf case, different from the quantum dimension of the corresponding bimodule (statistical dimension of the associated superselection sector). Consequently, the obstruction against non-integer quantum (or statistical) dimensions which seemed always inherent to C* symmetries, is absent for weak C* Hopf symmetries.

2 Weak C* Hopf algebras

The following defining system of axioms for weak C* Hopf algebras is due to ref. [4], to which we also refer for further details. Our emphasis here is on the departure from true C* Hopf algebras.

(A) A finite dimensional weak C* Hopf algebra is a C* algebra with $\mathbb{1}$ (hence a direct sum of matrix rings) with the additional structures coproduct, counit and antipode.

The coproduct is a coassociative $*$-homomorphism $\Delta\colon Q \to Q \otimes Q$. The counit is a positive linear map $\varepsilon\colon Q \to \mathbb{C}$ and satisfies the compatibility condition with the coproduct: $(\varepsilon \otimes id)\circ\Delta = id = (id \otimes \varepsilon)\circ\Delta$. The antipode is a complex-linear anti-homomorphism and anti-cohomomorphism $S\colon Q \to Q$ (i.e., it reverts the order of the product and of the coproduct), and is inverted by the $*$-structure: $S^{-1}(q) = S(q^*)^*$.

(B) The three axioms in the left column hold, as opposed to the corresponding three stronger axioms for true C* Hopf algebras in the right column.

$$\text{Weak: }\begin{cases} \Delta(\mathbb{1}) \equiv \mathbb{1}_{(1)} \otimes \mathbb{1}_{(2)} = \text{Projection} \\ \varepsilon(qp) = \varepsilon(q\mathbb{1}_{(1)}) \cdot \varepsilon(\mathbb{1}_{(2)}p) \\ S(q_{(1)})q_{(2)} \otimes q_{(3)} = (\mathbb{1} \otimes q) \cdot \Delta(\mathbb{1}) \end{cases} \qquad \text{True: }\begin{cases} \Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1} \\ \varepsilon(qp) = \varepsilon(q) \cdot \varepsilon(p) \\ S(q_{(1)})q_{(2)} = \varepsilon(q)\mathbb{1} \end{cases}$$

(We use the shorthand notation $q_{(1)} \otimes q_{(2)}$ for the expansion of $\Delta(q)$.) Along with the three weak axioms, either of the three stronger axioms implies the other two. Therefore these three axioms for true C* Hopf algebras cannot be independently relaxed.

(C) The dual $\widehat{Q}$ is defined by the linear maps $\{\hat{q}\colon Q \to \mathbb{C}\}$. The structure data of Q are canonically dualized by the pairing, and by $\langle \hat{q}^*, q \rangle = \overline{\langle \hat{q}, S(q)^* \rangle}$ the dual is given a $*$-structure. $\widehat{Q}$ with data $(\hat{\mathbb{1}} = \varepsilon, \hat{\Delta}, \hat{\varepsilon}, \hat{S})$ and $*$ is again a weak C* Hopf algebra.

(D) A (left) action of a weak C* Hopf algebra Q on a C* algebra B is a unital algebra homomorphism from Q into the linear maps of B into B, denoted by $b \mapsto (q \triangleright b)$, satisfying

$$\begin{matrix} q \triangleright (b \cdot c) = (q_{(1)} \triangleright b) \cdot (q_{(2)} \triangleright c), \\ (q^* \triangleright b^*)^* = S^{-1}(q) \triangleright b \end{matrix} \qquad \begin{pmatrix} q \in Q \\ b, c \in B \end{pmatrix}.$$

An element a of B is called invariant under the action, if $q \triangleright a = q_{(1)}S(q_{(2)}) \triangleright a$. The invariant elements form a $*$-subalgebra $A \equiv B^Q \subset B$.

3 Subfactors of depth 2

We consider a pair A, B of von Neumann factors of type III, along with an injective unit-preserving homomorphism $\iota\colon A \to B$. A may be thought of as a subfactor of B, and ι the inclusion map.

(A) [10] The subfactor $A \overset{\iota}{\hookrightarrow} B$ has finite index if and only if there is a homomorphism $\bar\iota\colon B \to A$ (the conjugate) and a "standard" pair of isometries $w \in A$, $v \in B$ satisfying

$$\begin{aligned} wa &= \varrho(a)w \qquad (a \in A), & vb &= \gamma(b)v \qquad (b \in B),\\ w^*\bar\iota(v) &= \lambda^{-\frac12}\cdot \mathbb{1}_A, & v^*\iota(w) &= \lambda^{-\frac12}\cdot \mathbb{1}_B, \end{aligned}$$

where $\gamma = \iota{\circ}\bar\iota$ and $\varrho = \bar\iota{\circ}\iota$ are the canonical and dual canonical endomorphisms of B and A, respectively, associated with the subfactor. $\lambda > 1$ is called the index of the subfactor associated with the pair w, v. In the following we assume w and v to minimize the index.

(B) [11] A "canonical triple" (ϱ, w, x) consists of an endomorphism ϱ of A and a pair of isometries w, x in A satisfying the relations (λ^2 being the index of $\varrho(A) \subset A$)

$$\begin{gathered} wa = \varrho(a)w, \qquad x\varrho(a) = \varrho^2(a)x \qquad (a \in A);\\ xx = \varrho(x)x, \quad xx^* = \varrho(x^*)x, \quad w^*x = \lambda^{-\frac12}\mathbb{1}_A = \varrho(w^*)x. \end{gathered}$$

These relations make certain that ϱ is a canonical endomorphism, i.e., one can draw a "conjugate square root" $\varrho = \bar\iota{\circ}\iota$ with w, v as in (A) and $x = \bar\iota(v)$.

Every such triple characterizes, up to isomorphism, an ambient algebra B such that

$$\varrho(A) \subset A_1 \subset A \overset{\iota}{\hookrightarrow} B$$

is a sequence of Jones extensions of index λ. Here A_1 is the intermediate algebra generated by $\varrho(A)$ and x, from which B is obtained by the Jones construction. (An additional non-redundancy condition on the triple prevents B from having a center. We shall tacitly always assume that B is a factor.) The minimal conditional expectation from B onto A is $\mu = w^*\bar\iota(\,\cdot\,)w$.

(C) [5] A dual pair of weak C* Hopf algebras is associated with every reducible subfactor of finite index λ and of depth 2. As algebras, these are the relative commutants $Q := \bar\iota{\circ}\iota(A)' \cap A$ and $\widehat{Q} := \iota{\circ}\bar\iota(B)' \cap B$. They are direct sums of matrix rings corresponding to the subsectors of $\bar\iota{\circ}\iota$ resp. $\iota{\circ}\bar\iota$ and their multiplicities, and have the same finite dimension.

Let z be the positive central element in the relative commutant $\iota(A)' \cap B$ with value $\sqrt{n_s/d_s}$ on each of the minimal central projections p_s of the relative commutant, where d_s^2 is the index of the corresponding reduced subfactor, and n_s is its multiplicity. Let $\bar z$ in the relative commutant $\bar\iota(B)' \cap A$ be defined analogously, and put $\overset{\bullet}{v} := zv \equiv \iota(\bar z)v$, $\overset{\bullet}{w} := \bar z w \equiv \bar\iota(z)w$, and $\overset{\circ}{v} := z^{-1}v \equiv \iota(\bar z^{-1})v$, $\overset{\circ}{w} := \bar z^{-1}w \equiv \bar\iota(z^{-1})w$.

The algebras Q and $\widehat{Q}$ are put into duality by the nondegenerate pairing

$$\langle \hat q, q\rangle := \lambda\cdot w^*\bar\iota(\overset{\bullet}{v})^* q\bar\iota(\hat q)\overset{\bullet}{w}w \equiv \lambda\cdot v^*\iota(\overset{\bullet}{w})^*\hat q\iota(q)\overset{\bullet}{v}v.$$

It induces a coproduct in Q (as a linear map into $Q\otimes Q$) from the product in $\widehat{Q}$. This coproduct is in fact a $*$-homomorphism if and only if the depth is 2, i.e., if and only if every subsector of $\iota{\circ}\bar\iota{\circ}\iota$ is already contained in ι.

The counit on Q is the pairing with the unit in $\widehat{Q}$. The antipode on Q is given by the two equivalent definitions

$$S(q) = \lambda\cdot\bar\iota\,(\iota\,[\overset{\circ}{w}{}^*\bar\iota(\overset{\bullet}{v})^*q]\,\overset{\bullet}{v})\,\overset{\circ}{w} \equiv \lambda\cdot\overset{\bullet}{w}{}^*\bar\iota\,(\overset{\circ}{v}{}^*\iota\,[q\bar\iota(\overset{\circ}{v})\overset{\bullet}{w}])$$

and is involutive up to a non-unitary conjugation:

$$S(S(q)) = [\bar z^{-1}\bar\iota(z)]^2\cdot q\cdot[\bar z^{-1}\bar\iota(z)]^{-2}.$$

The dual structures on $\widehat{Q}$ are given by the replacements $(A, w, \iota) \leftrightarrow (B, v, \bar{\iota})$. All statements here and in the remainder of this chapter are perfectly symmetric under this duality. Our emphasis will, however, be on those statements which pertain to the natural interpretation $A =$ fixpoints of B under a symmetry.

Proposition: *With the above definitions, Q and $\widehat{Q}$ are a dual pair of finite dimensional weak C* Hopf algebras. The dual formulae*

$$q \triangleright b := \lambda^{\frac{1}{2}} \cdot \mathring{v}^* \iota\,(q\bar{\iota}(b)\overset{\bullet}{w}) \qquad \text{and} \qquad \hat{q} \triangleright a := \lambda^{\frac{1}{2}} \cdot \mathring{w}^* \bar{\iota}\,(\hat{q}\iota(a)\overset{\bullet}{v})$$

define an action of Q on the algebra B with invariants $B^Q = \iota(A)$, and an action of $\widehat{Q}$ on A such that B is isomorphic with the crossed product of A by $\widehat{Q}$.

The last statement of the proposition means that $\widehat{Q}$ and A are embedded as subalgebras into B with relations

$$\hat{q} \cdot \iota(a) = \iota(\hat{q}_{(1)} \triangleright a) \cdot \hat{q}_{(2)} \qquad \text{or equivalently} \qquad \iota(\hat{q} \triangleright a) = \hat{q}_{(1)} \cdot \iota(a) \cdot \hat{S}(\hat{q}_{(2)}),$$

and together generate B.

(D) B is spanned by elements $\Gamma_e \iota(a)$ where $a \in A$ and $\Gamma_e \in B$ are intertwiners $\iota \circ \varrho_c \to \iota$ associated with every irreducible subsector ("charge") $\varrho_c \prec \bar{\iota} \circ \iota$. Under the action of Q the operators Γ_e of fixed charge transform as finite multiplets by matrix multiplication of the corresponding matrix ring of Q. The dimension of each multiplet equals the multiplicity of ϱ_c in $\bar{\iota} \circ \iota$.

Acting on B with the element $\mathring{w}\overset{\bullet}{w}^*$ of Q (which is related to the Haar measure) annihilates all multiplets of nontrivial charge and averages over those of trivial charge. The effect is precisely the minimal conditional expectation μ onto A.

(E) The action of Q on B is partly inner in the following sense. The relative commutant $\iota(A)' \cap B$ is mapped by $\bar{\iota}$ onto a subalgebra P of Q such that $S(P) \equiv \bar{P} = \bar{\iota}(B)' \cap A$. P and $\bar{P}$ are two commuting subalgebras of Q, and $P \cdot \bar{P}$ is a weak C* Hopf subalgebra of Q. One has $\Delta(p) = \Delta(\mathbb{1})(p \otimes \mathbb{1})\Delta(\mathbb{1})$ for $p \in P$ [4] and $\mathbb{1}_{(1)}\bar{\iota}(b)S(\mathbb{1}_{(2)}) = \bar{\iota}(\mathbb{1} \triangleright b) = \bar{\iota}(b)$, implying

$$\bar{\iota}(p\bar{p} \triangleright b) = p \cdot \bar{\iota}(b) \cdot S(\bar{p}) \qquad (p \in P, \bar{p} \in \bar{P}).$$

Since $P = S(\bar{P}) \subset \bar{\iota}(B)$ it follows that the action of $P \cdot \bar{P}$ on B is unitarily implemented by elements of B. This feature is not surprising since also for group actions it is well known that an inner action produces a reducible inclusion.

(F) [5] The action of Q on B preserves the subalgebra $\widehat{Q}$, and $\widehat{Q}$ preserves the subalgebra Q of A. The action of Q on $\widehat{Q}$ satisfies (and is determined by) the rule $\langle q \triangleright \hat{q}, p \rangle = \langle \hat{q}, pq \rangle$, and vice versa.

4 Application to the field problem

(A) A quantum field theory assigns to every bounded region $\mathcal{O}$ in space-time a weakly closed operator algebra $\mathcal{A}(\mathcal{O})$ generated by the fields (in a vacuum representation) localized in $\mathcal{O}$. The resulting isotonous net of algebras determines the theory, even without knowledge of the underlying fields. Under standard assumptions of covariance and spectrum condition, the local algebras are hyperfinite type III factors.

The theory is local if the local algebras associated with two regions at space-like distance commute with each other. A theory of observables has to be local.

The following results can be found in full detail in [12].

(B) A field extension of a theory of observables $\mathcal{A}$ is a (relatively local, but possibly nonlocal) isotonous net $\mathcal{B}$ such that for every region, $\mathcal{A}(\mathcal{O})$ is a subfactor of $\mathcal{B}(\mathcal{O})$. Assuming the existence of a conditional expectation $\mu\colon \mathcal{B} \to \mathcal{A}$ which preserves the localization and leaves the vacuum state invariant (an unbroken global symmetry in the broadest sense), the dual canonical endomorphism ϱ associated with a single local subfactor $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$ extends to a covariant endomorphism ϱ of the entire net of observables $\mathcal{A}$. The latter is trivial on observables at spacelike distance from $\mathcal{O}$, and is hence a DHR endomorphism [13] localized in $\mathcal{O}$. ϱ is equivalent to the representation of $\mathcal{A}$ on the vacuum Hilbert space of $\mathcal{B}$, and its subsectors are precisely those superselection charges of $\mathcal{A}$ for which there are charged fields within the ambient net $\mathcal{B}$.

(C) Conversely, every "DHR canonical triple" (ϱ, w, x) determines, in terms of observable data, a field extension $\mathcal{B}$ with an unbroken global symmetry. Here ϱ is a DHR endomorphism of $\mathcal{A}$ localized in some region $\mathcal{O}$, w and x are isometries in $\mathcal{A}(\mathcal{O})$, and the algebraic relations as in 3(B) hold with $a \in \mathcal{A}$.

The algebra $B \equiv \mathcal{B}(\mathcal{O})$ is constructed from the triple as in 3(B) with $A \equiv \mathcal{A}(\mathcal{O})$, and the other local algebras are obtained from it by Poincaré or conformal covariance. It is a nontrivial result about this construction that the structural properties of the "germinal" local subfactor $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$ propagate to every other local subfactor. In the sequel, statements about the net are understood to hold for every local subfactor.

The simple eigenvalue condition $\varepsilon_\varrho x = x$ on the statistics operator ε_ϱ of ϱ decides whether the net $\mathcal{B}$ resulting from the triple is local or not.

(D) A "cheap" way to obtain DHR canonical triples is the following. For any DHR endomorphism σ of $\mathcal{A}$ (with finite statistics) there is a standard pair (cf. 3(A)) of isometries w and $\bar{w}$ in $\mathcal{A}$ such that $wa = \sigma\bar{\sigma}(a)w$ and $\bar{w}a = \bar{\sigma}\sigma(a)\bar{w}$ for $a \in \mathcal{A}$. Thus $(\varrho = \sigma\circ\bar{\sigma}, w, x = \sigma(\bar{w}))$ is a DHR canonical triple. Therefore, every DHR sector σ defines a corresponding field extension $\mathcal{B}_\sigma$. If σ is localized in $\mathcal{O}$, then $B \equiv \mathcal{B}_\sigma(\mathcal{O})$ is the Jones extension of $A \equiv \mathcal{A}(\mathcal{O})$ by its subfactor $A_1 \equiv \sigma(\mathcal{A}(\mathcal{O}))$. The local subfactors have depth 2 if and only if all subsectors of $\sigma\bar{\sigma}\sigma$ are already contained in σ.

By the eigenvalue condition (cf. (C)), the nets $\mathcal{B}_\sigma$ are always nonlocal (unless σ is an automorphism and consequently $\mathcal{B}_\sigma = \mathcal{A}$). Since this statement is not in [12], we provide the argument here. One applies the standard left-inverse $\phi(a) = \bar{w}^*\bar{\sigma}(a)\bar{w}$ of σ to the eigenvalue condition. But $\phi(x) = \bar{w}$ is an isometry while, by the statistics calculus [3,13], $\phi(\varepsilon_{\sigma\bar{\sigma}}\sigma(\bar{w}))$ differs from an isometry by the statistics parameter of σ which is $1/d(\sigma)$ times a unitary. Therefore, equality can hold only if $d(\sigma) = 1$.

(E) This "cheap" construction does not provide all DHR canonical triples. Notably an irreducible extension with an outer action of a compact gauge group such that the observables are the fixed points cannot be of this type. E.g., provided all DHR sectors of $\mathcal{A}$ have permutation group statistics, the Doplicher-Roberts construction [14] determines a graded local extension with a compact gauge group. Its dual canonical endomorphism $\varrho = \varrho_{\mathrm{reg}}$ contains every irreducible DHR sector with multiplicity n_s equal to its statistical dimension d_s. The isometry x encodes the Clebsch-Gordan coefficients of the gauge group.

5 Putting things together

In a quantum field theory with superselection sectors, it is desirable to have a field algebra of charged fields which generate all charged sectors from the vacuum, and a gauge symmetry acting on the fields with the observables as fixpoints. In order to be of practical use, the transformation law for the fields should be sufficiently simple and concrete. These demands favour field extensions with weak C* Hopf symmetry which have transformation laws with finite multiplets, while in low dimensions it is in general inconsistent to ask for true C* Hopf symmetry.

(A) In order to find field extensions of a given (rational) local quantum field theory of observables $\mathcal{A}$ which have a weak C* Hopf symmetry, one has to look for DHR canonical triples (ϱ, w, x) such that the resulting local subfactors have depth 2.

(B) An immediate possibility in rational models is to choose $\sigma = \sigma_\oplus$ the direct sum of all irreducible DHR sectors of $\mathcal{A}$ with multiplicity one, and to proceed as in 4(D). By construction and since $\sigma_\oplus$ is self-conjugate, the resulting local subfactor $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$ is isomorphic to $\sigma_\oplus(\mathcal{A}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O})$. An explicit unitary equivalence shows [15] that the same holds for the reduced field bundle extension $\mathcal{B}_{\mathrm{red}}$. Therefore, the reduced field bundle equals the extension of $\mathcal{A}$ by $\sigma_\oplus$. It has depth 2 and a self-dual weak C* Hopf symmetry Q. As algebras, Q and $\widehat{Q}$ are both isomorphic to the relative commutant ${\sigma_\oplus}^2(\mathcal{A})' \cap \mathcal{A}$. The latter is a sum of matrix rings $M_s \cong \mathrm{Mat}_{N_s}(\mathbb{C})$ labelled by the DHR sectors ("charges") of $\mathcal{A}$, with N_s equal to their multiplicities within ${\sigma_\oplus}^2$.

The irreducible multiplets $\Gamma_e \iota(a)$ (cf. 3(D)) coincide with the charged operators $F_e(a)$ spanning the reduced field bundle as defined in [3]. Averaging over the symmetry by the conditional expectation μ (cf. 3(B,D)) yields the invariant elements $\sum_e F_e(a)$ where the sum extends over all edges of trivial charge. These are precisely the observables $\iota(a)$ on the extended Hilbert space.

(C) If all DHR sectors of $\mathcal{A}$ have integer statistical dimension, another natural choice is $\sigma = \sigma_{\mathrm{reg}} \equiv \varrho_{\mathrm{reg}}$ ($n_s = d_s$, $z = \mathbb{1} = \bar{z}$). This choice is possible whenever the observables are given as the fixed points of another net $\mathcal{B}$ under a finite gauge group, e.g., the graded local Doplicher-Roberts extension in the case of permutation group statistics, cf. 4(E).

In the diagram below, the first row is a Jones sequence due to the gauge symmetry, and so is the second row by definition of $\mathcal{B}_{\mathrm{reg}}$. Since an alternating subsequence of a Jones sequence is again a Jones sequence, the two obvious vertical equalities imply the third:

$$\begin{array}{ccccccccc} A_2 = \varrho_{\mathrm{reg}}(A) & \subset & A_1 & \subset & A & \subset & B & \subset & B_1 = B \rtimes G \\ \parallel\!| & & & & \parallel\!| & & & & \parallel \\ \sigma_{\mathrm{reg}}(A) & & \subset & & A & & \overset{\iota}{\hookrightarrow} & & B_{\mathrm{reg}} \end{array} \qquad \begin{pmatrix} A = \mathcal{A}(\mathcal{O}) \\ B = \mathcal{B}(\mathcal{O}) \end{pmatrix}.$$

It follows that the extension $\mathcal{B}_{\mathrm{reg}}$ of $\mathcal{A}$ by $\sigma_{\mathrm{reg}}(\mathcal{A})$ equals the extension of $\mathcal{B}$ by $\mathcal{A}$, which in turn equals the crossed product of $\mathcal{B}$ by its gauge group G.

(D) If all sectors of $\mathcal{A}$ are simple, then σ_{reg} coincides with $\sigma_\oplus$. There is an anyonic field extension $\mathcal{B}$ with an abelian symmetry group such that the dual canonical endomorphism is $\varrho_{\mathrm{reg}} = \sigma_{\mathrm{reg}} = \sigma_\oplus$. By combination of (B) and (C), the reduced field bundle is a crossed product of the anyonic extension by its abelian gauge group.

(E) In the general case with integer statistical dimensions, the Jones extension $\mathcal{B}_{\mathrm{reg}}$ of $\mathcal{A}$ by $\sigma_{\mathrm{reg}}(\mathcal{A})$ is to the same extent "larger" than $\mathcal{B}_{\mathrm{red}}$ as $\sigma_{\mathrm{reg}}(\mathcal{A})$ is "smaller" than $\sigma_\oplus(\mathcal{A})$. Clearly, one has no inclusion as algebras but rather a compression by an appropriate projection which selects $\sigma_\oplus \prec \sigma_{\mathrm{reg}}$.

(F) By combination of (C) and (E), if the sectors of $\mathcal{A}$ have permutation group statistics, then the reduced field bundle is a compression of the crossed product of the Doplicher-Roberts graded local field algebra by its gauge group.

6 Discussion

The paragroups assocciated with subfactors of finite index and depth 2 are weak C* Hopf algebras. Weak C* Hopf algebras therefore reconcile the generalized symmetry notions due to Ocneanu and due to Kac and Drinfel'd. They arise as symmetries in quantum field theoretical models whenever the local subfactors "observables $\subset$ fields" have depth 2 (the condition of finite index may presumably be relaxed).

Field extensions with global weak C* Hopf symmetries are encoded in terms of the observables by depth 2 DHR canonical triples (ϱ, w, x). An obvious and not very subtle class of such extensions, including the reduced field bundle, is described in 4(D). Relations between various such extensions are clarified. Notably if the observables are the fixed points under a finite non-abelian gauge group acting on a given field net, it is made clear in which sense the reduced field bundle exceeds the given field net (it corresponds to its crossed product by the gauge group), and in which sense it is smaller than the former (it is a compression which removes the gauge multiplicities).

For a given theory there may exist other (and more "economic" as far as the field problem is concerned) DHR canonical triples which are not of the form described in 4(D). The corresponding field extensions have a chance of being local or graded local and are therefore *a priori* more interesting than those "cheap" ones. Among them are the Doplicher-Roberts reconstruction in four dimensions, as well as some examples in chiral models (related to conformal embeddings) which generate only a subset of the superselection sectors of a given model. Since we know of no systematic way to construct such extensions, and notably we do not have a direct criterium on the canonical triple which ensures depth 2, we refrain from discussing this important issue in this report.

The weakness of the Hopf structure reflects the reducibility of the local inclusions of gauge invariants among the fields. According to 3(E), this feature is due to a nontrivial part of the quantum symmetry acting innerly on the field algebra. The implementing operators lie in the intersection of all local field algebras and commute with the observables. In the "regular" case 5(C) with a finite gauge group, they are the global implementers of the gauge group, and in the reduced field bundle case 5(B), they are the source and range projections going along with the charged operators [3].

These operators are redundant in order to solve the field problem, in the sense that they do not have any effect on the observables. Furthermore, they mix up local and global concepts, albeit on the unobservable level of charged fields. On the other hand, depth 2 and therefore a linear transformation law with finite dimensional symmetry tensors require their presence. We consider these peculiar field operators as the price to be paid for a decent symmetry acting on the charge carrying fields.

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