GENERALIZED CUNTZ ALGEBRAS ASSOCIATED WITH SUBFACTORS

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Abstract. Various generalizations of Cuntz algebras and their relations to symmetry and duality are reviewed. New generalized Cuntz algebras are associated with a subfactor. A characteristic Hilbert space of basic invariants (with respect to the generalized symmetry) within these algebras is discussed.

1. Motivation

We consider an irreducible inclusion $A \subset B$ of properly infinite von Neumann factors, and denote by $\iota \in Hom(A,B)$ the inclusion map $A \ni a \mapsto a \in B$. We assume the index to be finite. Then there is a conjugate endomorphism $\bar{\iota}: B \to A$ such that $\iota\circ\bar{\iota} \in End(B)$ is the canonical endomorphism of B into A, and $\bar{\iota}\circ\iota \in End(A)$ is its dual [6]. We are interested in the irreducible subsectors ϱ of $\bar{\iota}\circ\iota$. If $\varrho \prec \bar{\iota}\circ\iota$, then equivalently $\iota \prec \iota\circ\varrho$, that is, there exist intertwiners ψ in B which satisfy

$$\psi \cdot \iota(a) = \iota\varrho(a) \cdot \psi \qquad (a \in A)\ . \tag{1.1}$$

The existence of nontrivial such intertwiners, in turn, characterizes the subsectors of $\bar{\iota}\circ\iota$.

In a specific quantum field theoretical context, A and B may be thought of as a pair of local algebras of observables and of charged fields, respectively. Under standard physical assumptions ("unbroken symmetry", cf. [8]) the dual canonical endomorphism $\bar{\iota}\circ\iota$ of A extends to a localized endomorphism of the C*-algebra of observables, and can be identified with the restriction of the vacuum representation of the field algebra to the observables [8]. The intertwiners (1.1) are then field operators interpolating between the vacuum representation of the observables and a charged representation corresponding to ϱ. We shall therefore call these intertwiners "charged operators" also in general. The charged operators form a Hilbert space H of isometries within B.

If $A \subset B$ has depth 2, then A are the fixpoints under the action of a Hopf algebra (e.g., a group) G on B, and the Hilbert space of charged operators for a given subsector ϱ has support projection $\mathbb{1}_B$ in B and carries a unitary representation of G [7]. This establishes a correspondence between subsectors of $\bar{\iota}\circ\iota$ and representations of the symmetry. The duality problem amounts to the reconstruction of G and B from the knowledge of the relevant endomorphisms $\varrho \in End(A)$ only [3,4].

One way to do so (if G is a finite group) is to construct, for suitable ϱ of dimension d, the Cuntz algebra $\mathcal{O}(H) \cong \mathcal{O}_d$ generated by a d-dimensional Hilbert space H of isometries of unit support [2], and to extend A by $\mathcal{O}(H)$ by postulating the commutation relations (1.1) for $\psi \in H$. One thus obtains a "crossed product by the action of the endomorphism". In order to recover B, however, a certain subalgebra of the Cuntz algebra has to be identified with the

subalgebra $\mathcal{O}_\varrho$ of A generated by the intertwiners $T \in A$ between powers of ϱ, that is

$$T \cdot \varrho^n(a) = \varrho^m(a) \cdot T \qquad (a \in A)\ . \tag{1.2}$$

The unitary transformations of H induce automorphisms of $\mathcal{O}(H)$. The identification of the subgroup $G \subset U(H)$ (the symmetry group) which has $\iota(\mathcal{O}_\varrho)$ as its fixpoints within $\mathcal{O}(H)$, constitutes the crucial step in the reconstruction problem [3].

We want to understand the role of the charged operators for the position of $\iota(A)$ in B in the case of arbitrary depth, i.e., not given by the f ixpoints under a Hopf algebra. One could again consider the C*-subalgebra of B generated by the charged operators, and study its relation with the C*-algebra $\mathcal{O}_\varrho \subset A$ generated by all elements satisfying (1.2) for some n, m.

But the latter algebra, considered as a subalgebra of B, is in general not contained in the former. It is therefore desirable to identify a larger C*-algebra $\mathcal{O}_{\iota,\varrho}$ containing the charged operators as well as the invariants $\mathcal{O}_\varrho$. In order to keep this algebra minimal, we require that its image under the unique normal conditional expectation $\mu\colon B \to A$ gives exactly $\mathcal{O}_\varrho$:

$$\mu(\mathcal{O}_{\iota,\varrho}) = \mathcal{O}_\varrho \qquad \text{and} \qquad \iota(\mathcal{O}_\varrho) \subset \mathcal{O}_{\iota,\varrho}\ . \tag{1.3}$$

The general idea is to consider the map $\mu\colon \mathcal{O}_{\iota,\varrho} \to \mathcal{O}_\varrho$ as a "model action" for the generalized symmetry underlying $\mu\colon B \to A$.

This is the setting of our present analysis. We shall assume that ι and ϱ have finite index, and that both ϱ^n and $\iota\circ\varrho^n$, as n varies, generate only finitely many irreducible subsectors ("rationality"). On the other hand, we drop the assumption that ι is contained in $\iota\circ\varrho$ (or $\varrho \prec \bar{\iota}\circ\iota$), that is, possibly there are no charged operators in the proper sense. By rationality, however, ι will be contained in $\iota\circ\varrho^n$ for some n, so there will always be "higher" charged operators for ϱ^n.

Besides, A need not be a subfactor of B, but rather ι may be any irreducible homomorphism between two properly infinite factors. Identifying A with its image $\iota(A)$ in B, the former situation will always be recovered.

With these data we associate two C*-algebras, namely $\mathcal{O}_\varrho \subset A$ which is generated by all operators in A which intertwine powers of $\varrho \in End(A)$ as before, and $\mathcal{O}_{\iota,\varrho} \subset B$ which is generated by all operators in B which intertwine powers of ϱ (for the image of A in B like in (1.1)).

In order to analyze these generalized Cuntz algebras, we study in both algebras the linear subspace of intertwiners between $\mathrm{id}_A = \varrho^0$ and ϱ^n, and identify within this subspace a generating Hilbert space of isometries ("skeleton space"). Surprisingly, in some exceptional cases the skeleton space is finite-dimensional and has unit support. In these cases, it actually generates the entire generalized Cuntz algebra. In general, the skeleton space is infinite-dimensional, but its support always converges to unity in a natural Hilbert norm.

These observations give rise to embeddings of ordinary Cuntz algebras (generated by the skeleton spaces) into generalized Cuntz algebras. While *a priori* such embeddings are nothing peculiar, the ones due to the skeleton space are of some relevance to the motivating duality problem, since the skeleton space of $\mathcal{O}_{\iota,\varrho}$ collects the algebraically independent "higher" charged operators.

The Cuntz algebras generated by the skeleton spaces are in general dense within the generalized Cuntz algebras (in an appropriate topology). In the exceptional cases with skeleton spaces of finite dimension, there is equality, that is, $\mathcal{O}_{\iota,\varrho}$ may in fact turn out to be the ordinary Cuntz algebra generated by its skeleton space.

As a byproduct, we give a simple characterization of the exceptional cases in terms of fusion matrices.

There is a certain overlap with work done by Izumi [5]. In particular, the A_4-example described in Sect. 3 can be found also in that paper, although considered from a different angle of view. We are not aware of a relation to another recent generalization of Cuntz algebras due to Pimsner [9].

2. Generalized Cuntz algebras

The Cuntz algebras $\mathcal{O}_d$ were introduced in [2] as the C*-algebras generated by a (possibly countably-infinite) number $d \geq 2$ of orthogonal isometries v_i, subject to the relation $\sum_i v_i v_i^* = \mathbb{1}$ if $d < \infty$. They were shown to be simple C*-algebras, depending only on d up to isomorphisms. We are here mainly interested in concrete realizations of Cuntz algebras on a Hilbert space $\mathcal{H}$.

2.1 Definition. *Let $H \subset \mathcal{B}(\mathcal{H})$ be a Hilbert space of isometries, that is $v^* w$ is a multiple of $\mathbb{1}_{\mathcal{H}}$ for $v, w \in H$. Then $\mathcal{O}(H) \subset \mathcal{B}(\mathcal{H})$ is the C*-algebra generated by the elements of H:*

$$\mathcal{O}(H) := \left[\bigcup_{n,m \geq 0} H^m (H^n)^* \right]^{C^*} \subset \mathcal{B}(\mathcal{H}) \ .$$

The abelian case $\dim H = 1$ is of no interest. If H is separable, or if $2 \leq \dim H < \infty$ and H has unit support, then $\mathcal{O}(H)$ is isomorphic with the Cuntz algebra $\mathcal{O}_{\dim H}$, by identification of v_i with an orthonormal basis of H. In contrast, if H is finite-dimensional but of support $< \mathbb{1}$, then $\mathcal{O}(H)$ is known as a Cuntz-Toeplitz algebra which has a nontrivial ideal generated by $\sum_i v_i v_i^* - \mathbb{1}$.

It was observed in [3] in a new approach to duality for compact groups, that $\mathcal{O}_\infty$ is not the appropriate object for duality theory. The authors rather conceptualized the definition by viewing the generating Hilbert space H as an object of the category of Hilbert spaces. This leads to a different generalization to the infinite-dimensional case. Their general definition is

2.2 Definition. *Let X be an object of a strict C* tensor category. The spaces of arrows between tensor (= monoidal) powers of X are denoted by $(X^{\times n}, X^{\times m})$. These spaces are embedded into $(X^{\times n+1}, X^{\times m+1})$ by taking the right tensor (= monoidal) product with $\mathbb{1}_X$. Then*

$$\mathcal{O}_X := \left[\bigcup_{n,m \geq 0} (X^{\times n}, X^{\times m}) \right]^{C^*} \ .$$

Example 2.1: The category of Hilbert spaces is a strict C* tensor category where the arrows (H_1, H_2) are given by the homomorphisms $(H_1, H_2) =$

$Hom(H_1, H_2)$. For a finite-dimensional Hilbert space H of unit support, $\mathcal{O}_H$ is isomorphic with $\mathcal{O}(H)$ by the natural identification of $(H^{\otimes n}, H^{\otimes m}) \subset \mathcal{O}_H$ with $H^m(H^n)^* \subset \mathcal{O}(H)$:

$$\mathcal{O}_H \cong \mathcal{O}(H) \cong \mathcal{O}_{\dim H} \ . \tag{2.1}$$

In contrast, for H infinite-dimensional, $\mathcal{O}(H)$ is naturally embedded as a proper subalgebra into $\mathcal{O}_H$. The reason is, in a nutshell, that $(H, H) \subset \mathcal{O}_H$ is isomorphic with $\mathcal{B}(H)$, while its intersection HH^* with $\mathcal{O}(H)$ contains only the compact operators.

Example 2.2: The category of unitary representations $D\colon G \to \mathcal{B}(H)$ of a compact group (or Hopf algebra) G is a strict C* tensor category where the arrows (D_1, D_2) are the intertwiners $Hom_G(H_1, H_2)$. If D is a representation of G on the finite-dimensional Hilbert space H of isometries, then $(D^{\otimes n}, D^{\otimes m})$ is naturally identified with the invariants under the induced action of G on $H^m(H^n)^* \subset \mathcal{O}(H)$, hence

$$\mathcal{O}_D \cong \mathcal{O}(H)^G \ . \tag{2.2}$$

Of particular interest for us is the case when $X = \varrho$ is a unital endomorphism of some properly infinite von Neumann factor A. The endomorphisms of a properly infinite factor form a strict C* tensor category, where the arrows are the intertwiners,

$$(\varrho_1, \varrho_2) = \{T \in A : T\varrho_1(a) = \varrho_2(a)T \quad \forall a \in A\} \qquad (\varrho_1, \varrho_2 \in End(A)) \ . \tag{2.3}$$

The following definition is just a special case of Definition 2.2. We assume ϱ to be proper, that is, not an automorphism.

2.3 Definition. *Let ϱ be a proper unital endomorphism of A. Then*

$$\mathcal{O}_\varrho := \left[\bigcup_{n,m \geq 0} (\varrho^n, \varrho^m) \right]^{C^*} \subset A \ .$$

Since ϱ maps (ϱ^n, ϱ^m) into $(\varrho^{n+1}, \varrho^{m+1})$, its action on A preserves the subalgebra $\mathcal{O}_\varrho$.

Example 2.3: If ϱ is inner, i.e., $\varrho(a) = \sum_i v_i a v_i^*$ with isometries $v_i \in A$, then

$$\mathcal{O}_\varrho = \mathcal{O}(H) \ , \tag{2.4}$$

where H is the Hilbert space spanned by v_i with inner product $(v, w) = v^*w$. As an endomorphism of $\mathcal{O}(H)$, ϱ is the canonical inner endomorphism associated with H.

We now introduce a new generalized Cuntz algebra, denoted $\mathcal{O}_{\iota,\varrho}$, which is associated with an irreducible homomorphism $\iota\colon A \to B$ between two properly infinite von Neumann factors A and B, and a unital endomorphism ϱ of A as before. The intertwiners for homomorphisms between properly infinite factors are given by

$$(\sigma_1, \sigma_2) = \{T \in B : T\sigma_1(a) = \sigma_2(a)T \quad \forall a \in A\} \qquad (\sigma_1, \sigma_2 \in Hom(A, B)) \ . \tag{2.5}$$

2.4 Definition. *Let ι be an irreducible unital homomorphism of A into B, and ϱ a proper unital endomorphism of A. Then*

$$\mathcal{O}_{\iota,\varrho} := \left[\bigcup_{n,m\geq 0} (\iota\circ\varrho^n, \iota\circ\varrho^m) \right]^{C^*} \subset B \ .$$

If μ is a conditional expectation from B onto $\iota(A)$, then μ maps $(\iota\circ\varrho^n, \iota\circ\varrho^m)$ onto $\iota(\varrho^n, \varrho^m)$, hence

$$\mu(\mathcal{O}_{\iota,\varrho}) = \iota(\mathcal{O}_\varrho) \ . \tag{2.6}$$

The special case $B = A$, $\iota = \mathrm{id}_A$ yields $\mathcal{O}_{\iota,\varrho} = \mathcal{O}_\varrho$, of course. Our original motivation concerns the case $A \subset B$ and ι the inclusion map $A \ni a \mapsto a \in B$, and ϱ an irreducible subsector of $\bar\iota\circ\iota$. Then equivalently $\iota \prec \iota\circ\varrho$, that is, the linear subspace of charged operators in B

$$H_{\iota,\varrho} = (\iota, \iota\circ\varrho) = \{\psi \in B : \psi a = \varrho(a)\psi \quad \forall a \in A\} \tag{2.7}$$

is a non-trivial Hilbert space of isometries contained in $\mathcal{O}_{\iota,\varrho}$.

Example 2.4: Let $A \subset B$ irreducible be the fixpoints under a Hopf algebra G, and $\varrho \prec \bar\iota\circ\iota$. Then $H_{\iota,\varrho}$ has unit support in B and is stable under the action of G and therefore carries a unitary representation D of G [7]. Furthermore, as an endomorphism of A, ϱ is implemented within B by $H_{\iota,\varrho} \subset B$,

$$\varrho(a) = \sum_i \psi_i \ a \ \psi_i^* \qquad \forall a \in A \ . \tag{2.8}$$

One has

$$\mathcal{O}_{\iota,\varrho} = \mathcal{O}(H_{\iota,\varrho}) \cong \mathcal{O}_{H_{\iota,\varrho}} \cong \mathcal{O}_d \tag{2.9}$$

where $d = d(\varrho) = dim(H_{\iota,\varrho}) = dim(D)$. The generalized Cuntz algebra $\mathcal{O}_\varrho \subset A$ equals the fixpoints of $\mathcal{O}_{\iota,\varrho} \subset B$ under the Hopf algebra action. In view of Example 2.2, $\mathcal{O}_\varrho$ can be identified with $\mathcal{O}_D$:

$$\mathcal{O}_\varrho = \mathcal{O}_{\iota,\varrho} \cap A = [\mathcal{O}_{\iota,\varrho}]^G \cong \mathcal{O}_D \ . \tag{2.10}$$

$\mathcal{O}_\varrho$ may even be isomorphic with $\mathcal{O}_{\iota,\varrho}$ itself, namely when $\varrho = \bar\iota\circ\iota$ and D is the regular representation [1,5,7]; but in general it cannot be expected to be of the form $\mathcal{O}(H)$ for any Hilbert space H.

In general, the support of $H_{\iota,\varrho}$ is a nontrivial projection $< \mathbb{1}_B$, and $\mathcal{O}_{\iota,\varrho}$ is not generated by the Hilbert space $H_{\iota,\varrho}$. Because the dimensions of endomorphisms are multiplicative and additive, and because $\dim(H_{\iota,\varrho})$ equals the multiplicity of ι within $\iota\circ\varrho$, one always has $\dim(H_{\iota,\varrho}) \leq d(\varrho)$. But our assumptions even admit the possibility $H_{\iota,\varrho} = \{0\}$.

The previous examples are rather special illustrations. We emphasize that they do not exhaust the classes of generalized Cuntz algebras of type $\mathcal{O}_\varrho$ and $\mathcal{O}_{\iota,\varrho}$.

3. The skeleton space

We consider $\mathcal{O}_{\iota,\varrho}$ as defined in Section 2. (The special case $B = A$, $\iota = \mathrm{id}_A$ covers also $\mathcal{O}_\varrho$.) We assume ι and ϱ to be of finite dimension, that is $\iota(A) \subset B$ and $\varrho(A) \subset A$ are subfactors of finite index.

For each $n \in \mathbb{N}_0$, $\mathcal{O}_{\iota,\varrho}$ contains the finite-dimensional subspace $h_n = (\iota, \iota\circ\varrho^n)$. These are Hilbert spaces of isometries since $h_n^* h_n \subset (\iota,\iota) = \mathbb{C}$. Being subspaces of B, h_n can be multiplied, and $h_n h_m \subset h_{n+m}$. Accordingly, let h'_n denote the span of all subspaces $h_{n_1}\cdots h_{n_r}$ with $1 \le n_i < n$ and $\sum_i n_i = n$ within h_n, and k_n the orthogonal complement of h'_n in h_n. Trivially, $h_0 = \mathbb{C}$, $k_1 = h_1 = H_{\iota,\varrho}$. Then

3.1 Lemma. *$x^*y = 0$ for $x \in k_n$, $y \in k_m$, provided $m \neq n$ and $n, m \neq 0$.*

Proof. Without loss of generality we may assume $m > n$. Then $x^*y \in h_{m-n}$. It follows that $x \cdot x^*y \in h_n h_{m-n} \subset h'_m$, and by definition this space is orthogonal to k_m, hence $y^*(x\cdot x^*y) = (x^*y)^*(x^*y) = 0$. □

3.2 Definition. *The skeleton space associated with $\mathcal{O}_{\iota,\varrho}$ is the Hilbert space*

$$K_{\iota,\varrho} := \bigoplus_n k_n \subset \mathcal{O}_{\iota,\varrho}\ .$$

For $B = A$, $\iota = id_A$, we call $K_\varrho \equiv K_{\iota,\varrho}$ the skeleton space associated with $\mathcal{O}_\varrho \equiv \mathcal{O}_{\iota,\varrho}$.

The obvious inclusions $H_{\iota,\varrho} \subset K_{\iota,\varrho} \subset \mathcal{O}(K_{\iota,\varrho}) \subset \mathcal{O}_{\iota,\varrho}$ hold.

We introduce the power series

$$h(t) = \sum_{n\ge 0} \dim(h_n) t^n \qquad \text{and} \qquad k(t) = \sum_{n>0} \dim(k_n) t^n \tag{3.1}$$

as generating functionals for the dimensions of h_n and k_n. By construction, h_n are generated by the subspaces $k_{n_1}\cdots k_{n_r}$ for all partitions $\sum n_i = n$. These are mutually orthogonal by Lemma 3.1. A little combinatorics therefore yields

3.3 Lemma. *The generating functionals for the dimensions of h_n and k_n are related by*

$$h(t) = \frac{1}{1-k(t)} \qquad \Longleftrightarrow \qquad k(t) = 1 - \frac{1}{h(t)}\ .$$

Next, let us assume that the family of irreducible subsectors σ_i of $\iota\circ\varrho^n$ as n varies, is finite ("rationality"). This property holds, e.g., if $\iota(A) \subset B$ is a subfactor of finite depth and ϱ is contained in some power of $\bar\iota\circ\iota$, that is, ϱ is a vertex in the even subgraph of the principal graph associated with the subfactor. The assumption is also satisfied if $B = A$, $\iota = \mathrm{id}$, and ϱ is an object of a category of endomorphisms with only finitely many irreducible objects, e.g., the superselection sectors of a "rational" quantum field theory. We then have a finite fusion matrix $(N_{\iota,\varrho})_{ij}$ given by the non-negative multiplicities of σ_i within $\sigma_j\circ\varrho$, where σ_i range over all irreducible subsectors of $\iota\circ\varrho^n$ and one of the indices $(i = \iota)$ corresponds to $\sigma_\iota \equiv \iota$. Since $\dim(h_n) = [(N_{\iota,\varrho})^n]_{\iota\iota}$, one obtains the formula for the generating functional

$$h(t) = [(1 - tN_{\iota,\varrho})^{-1}]_{\iota\iota}\ , \tag{3.2}$$

from which $k(t)$ can be obtained with the help of the Lemma 3.3. Explicitly, if P is the projection matrix onto the ι-direction and $Q = \mathbb{1} - P$, one has

$$\dim(k_n) = [N_{\iota,\varrho} Q N_{\iota,\varrho} \dots N_{\iota,\varrho} Q N_{\iota,\varrho}]_{\iota\iota} \equiv [N_{\iota,\varrho} (Q N_{\iota,\varrho} Q)^{n-2} N_{\iota,\varrho}]_{\iota\iota} \ . \qquad (3.3)$$

Clearly, $\dim(K_{\iota,\varrho}) = k(1)$.

It is well known that the largest eigenvalue of $N_{\iota,\varrho}$ is given by the dimension $d(\varrho)$ of ϱ, and the eigenvector is the Frobenius eigenvector $F_i = d(\sigma_i)$. It follows that the radius of convergence of the generating functional $h(t)$ is $1/d(\varrho) < 1$.

Example 3.1: Let $\varrho \in End(A)$ be implemented by a Hilbert space $H \in B$. Then $h_1 = H$, $h_n = H^n = h_n'$ and $k_1 = H$, $k_n = \{0\}$ $(n > 1)$, hence $K_{\iota,\varrho} = H$. This yields $h(t) = (1 - dt)^{-1}$ where $d = \dim(H) = d(\varrho)$, and $k(t) = dt$. As mentioned before, $\mathcal{O}_{\iota,\varrho} = \mathcal{O}(K_{\iota,\varrho}) = \mathcal{O}(H)$. Extending ϱ to an inner endomorphism σ of B, we have $K_\sigma = K_{\iota,\varrho}$ and $\mathcal{O}_\sigma = \mathcal{O}_{\iota,\varrho} = \mathcal{O}(H)$.

Example 3.2: On the other hand, the algebra $\mathcal{O}_\varrho$ associated with $\varrho \in End(A)$ from Example 3.1 will in general have a skeleton space of infinite dimension. If, for instance, A are the fixpoints of B under a non-abelian symmetry group G and ϱ corresponds to a representation D of G, then this space is determined by the representation theory of G; namely a basis of $k_n \subset K_\varrho$ corresponds to the independent G-invariant tensors within $D^{\otimes n}$. We display the generating functional $k(t)$ for $G = S_3$ and ϱ corresponding to the two-dimensional representation: $k(t) = t^2/(1 - t - t^2)$.

Example 3.3: Let $\iota(A) \subset B$ be a subfactor with principal graph A_4. Then $\bar\iota\circ\iota \cong \mathrm{id}_A \oplus \varrho$ with $d(\varrho) = \frac{1+\sqrt{5}}{2}$. The relevant fusion rules are $\iota\circ\varrho \cong \iota \oplus \alpha$ where α is an isomorphism of A with B, and $\alpha\circ\varrho \cong \iota$. The above prescription yields

$$h(t) = [(1 - tN_{\iota,\varrho})^{-1}]_{\iota\iota} = \frac{1}{1 - t - t^2} \qquad \text{and} \qquad k(t) = t + t^2 \ .$$

In other words, k_1 and k_2 are one-dimensional, and all higher k_n are trivial.

Example 3.4: Consider now $\mathcal{O}_\varrho$ with ϱ as in Example 3.3. In this case, the relevant fusion rules are $\mathrm{id}\circ\varrho = \varrho$ and $\varrho^2 \cong \mathrm{id} \oplus \varrho$, yielding

$$h(t) = [(1 - tN_\varrho)^{-1}]_{00} = \frac{1 - t}{1 - t - t^2} \qquad \text{and} \qquad k(t) = \frac{t^2}{1 - t} \ .$$

This shows that K_ϱ is infinite-dimensional.

We can look at the Examples 3.3, 3.4 more explicitly. Namely, due to the "Lee-Yang" fusion rules $\varrho^2 \cong \mathrm{id} \oplus \varrho$, there are isometries $v_1 \in (\varrho, \varrho^2)$ and $v_2 \in (\mathrm{id}, \varrho^2)$ in A satisfying $v_1 v_1^* + v_2 v_2^* = \mathbb{1}_A$. Actually, this pair of isometries generates $\mathcal{O}_\varrho$, so $\mathcal{O}_\varrho$ is isomorphic with the ordinary Cuntz algebra $\mathcal{O}_2$. But only v_2 is contained in K_ϱ.

To study $\mathcal{O}_{\iota,\varrho}$, one may choose α within its inner equivalence class such that $\alpha\circ\varrho = \iota$. It follows that $\psi_i = \alpha(v_i)$ are isometries $\psi_1 \in (\iota, \iota\circ\varrho)$ and $\psi_2 \in (\alpha\circ\varrho, \alpha\circ\varrho^3) = (\iota, \iota\circ\varrho^2)$ *a forteriori*. This pair of isometries is a basis of $K_{\iota,\varrho}$. Trivially, $\iota(\mathcal{O}_\varrho)$ is (properly) contained in $\mathcal{O}_{\iota,\varrho}$, while we shall see below that $\mathcal{O}_{\iota,\varrho}$ is in fact generated by ψ_i, that is $\mathcal{O}_{\iota,\varrho} = \alpha(\mathcal{O}_\varrho)$. Thus we have the proper inclusion

$$\iota(\mathcal{O}_\varrho) \subset \mathcal{O}_{\iota,\varrho} = \alpha(\mathcal{O}_\varrho) \ . \qquad (3.4)$$

We see in these examples that the skeleton space is not an invariant under isomorphisms of generalized Cuntz algebras, but depends on the presentation in terms of ι and ϱ.

4. Density

We are concerned with the question how the skeleton space $K_{\iota,\varrho}$ and the Cuntz algebra $\mathcal{O}(K_{\iota,\varrho})$ generated by this Hilbert space of isometries are embedded into $\mathcal{O}_{\iota,\varrho}$. Before we consider the general case, we return to the Examples 3.3, 3.4 associated with the A_4 inclusion, discussed in the previous section. It exhibits the general mechanism in a most transparent manner.

We therefore assume, as in Example 3.3, $\iota \in Hom(A,B)$ with the fusion rules $\bar{\iota}\circ\iota \cong \mathrm{id} \oplus \varrho$, $\iota\circ\varrho \cong \iota \oplus \alpha$, $\alpha\circ\varrho = \iota$. Let $X \in (\iota\circ\varrho^n, \iota\circ\varrho^m) \in \mathcal{O}_{\iota,\varrho}$. Then X is a linear combination of operators $T_a T_b^*$ and $S_c S_d^*$ with isometries $T_a \in (\iota, \iota\circ\varrho^m), T_b \in (\iota, \iota\circ\varrho^n)$ and $S_c \in (\alpha, \iota\circ\varrho^m), S_d \in (\alpha, \iota\circ\varrho^n)$. Since $(\iota, \iota\circ\varrho^n) = h_n$ and $(\alpha, \iota\circ\varrho^n) \subset (\alpha\circ\varrho, \iota\circ\varrho^{n+1}) = h_{n+1}$, and since in turn h_n are generated by k_1 and k_2 which span $K_{\iota,\varrho}$, we conclude that X is in the Cuntz algebra generated by $K_{\iota,\varrho}$, that is, $\mathcal{O}_{\iota,\varrho} = \mathcal{O}(K_{\iota,\varrho}) \cong \mathcal{O}_2$.

The situation is different for the Example 3.4. The endomorphism ϱ is the same as before, but $\mathcal{O}_\varrho$ is generated by intertwiners between powers of ϱ only. So let $X \in (\varrho^n, \varrho^m) \in \mathcal{O}_\varrho$. Similar as before, X is a linear combination of operators $R_a R_b^*$ and $U_c U_d^*$ with isometries $R_a \in (\mathrm{id}, \varrho^m) = h_m, R_b \in (\mathrm{id}, \varrho^n) = h_n$ and $U_c \in (\varrho, \varrho^m), U_d \in (\varrho, \varrho^n)$. This time, the latter can not be directly identified with elements of K_ϱ. But (ϱ, ϱ^n) is embedded into $(\varrho^2, \varrho^{n+1})$ and $\varrho^2 \cong \mathrm{id} \oplus \varrho$ is decomposed with the pair of isometries v_1 and v_2. Thus we may write

$$U_c U_d^* = U_c v_2 v_2^* U_d^* + U_c v_1 v_1^* U_d^* \ , \tag{4.1}$$

where $U_c v_2 \in h_{m+1}, U_d v_2 \in h_{n+1}$. Hence the first term in the decomposition is in the Cuntz algebra $\mathcal{O}(K_\varrho)$. The second term is of the form $\tilde{U}_c \tilde{U}_d^*$ with $\tilde{U}_c \in (\varrho, \varrho^{m+1}), \tilde{U}_d \in (\varrho, \varrho^{n+1})$, and can be treated iteratively like $U_c U_d^*$ before. Thus we end up with a decomposition

$$U_c U_d^* = \sum_{r=0}^{R} (U_c v_1^r v_2)(U_d v_1^r v_2)^* + (U_c v_1^{R+1})(U_d v_1^{R+1})^* \tag{4.2}$$

where the terms in the sum are in $h_{m+r+1} h_{n+r+1}^*$, hence are generated by K_ϱ. We shall show that the remainder term converges to zero in an appropriate topology (but not in the operator norm), as R increases.

In order to do so, we recall some definitions pertaining to the general case. For σ a unital homomorphism of A into B of finite dimension, the standard left-inverse ϕ_σ is the positive unital map $\sigma^{-1}\circ E_\sigma$ where E_σ is the minimal conditional expectation of B onto its subfactor $\sigma(A)$. Explicitly, ϕ_σ is of the form $\phi_\sigma(x) = R^* \bar{\sigma}(x) R$ where $\bar{\sigma}$ is a conjugate endomorphism of σ and R is an isometry in $(\mathrm{id}_A, \bar{\sigma}\sigma)$. By construction, ϕ_σ satisfies $\phi_\sigma(\sigma(x) y \sigma(z)) = x \phi_\sigma(y) z$ and in particular $\phi_\sigma \circ \sigma = \mathrm{id}_A$. Standard left-inverses are related by the "intertwining property"

$$d(\sigma_1)\phi_{\sigma_1}(xT) = d(\sigma_2)\phi_{\sigma_2}(Tx) \qquad \text{if} \qquad T \in (\sigma_1, \sigma_2) \ . \tag{4.3}$$

We consider the state on $\mathcal{O}_\varrho$ given by

$$\varphi_\varrho = \lim_{N\to\infty} \phi_\varrho^N \ . \tag{4.4}$$

ϕ_ϱ maps (ϱ^n, ϱ^m) into $(\varrho^{n-1}, \varrho^{m-1})$ as long as $n, m > 0$. Hence, its repeated application takes (ϱ^n, ϱ^n) eventually to the scalars $(\mathrm{id}_A, \mathrm{id}_A) = \mathbb{C}$ where further application of ϕ_ϱ is trivial. On the other hand, if $n \neq m$, and without loss of generality $n < m$, then (ϱ^n, ϱ^m) is mapped by ϕ_ϱ^n to the Hilbert space h_{m-n}. This space is mapped by ϕ_ϱ onto itself, and it can be shown that the restriction of ϕ_ϱ to h_k $(k > 0)$ is a bounded map with norm $\leq d(\varrho)^{-1} < 1$. Hence the limit $\varphi_\varrho = \lim_{N\to\infty} \phi^N$ stabilizes on the homogeneous part of $\mathcal{O}_\varrho$ and converges to zero on the inhomogeneous part of $\mathcal{O}_\varrho$. In other words, φ_ϱ is a state on $\mathcal{O}_\varrho$ which is invariant under the automorphism group on $\mathcal{O}_\varrho$ defined by $\alpha_t(x) = e^{i(n-m)t}x$ if $x \in (\varrho^n, \varrho^m)$. The intertwining property (4.3) implies

$$\varphi_\varrho(xy) = d(\varrho)^{n-m}\varphi_\varrho(yx) \qquad \text{for} \qquad x \in (\varrho^n, \varrho^m)\ , \tag{4.5}$$

that is, φ_ϱ is actually a KMS state for this automorphism group with temperature $2\pi/\ln d(\varrho)$.

The intertwining property allows to compute the expectation values

$$\varphi_\varrho(TT^*) = \frac{d(\tau)}{d(\varrho)^n} \tag{4.6}$$

whenever $T \in (\tau, \varrho^n)$ is an isometry and $\tau \prec \varrho^n$ is an irreducible subsector of ϱ^n.

Similarly, we consider the state on $\mathcal{O}_{\iota,\varrho}$ given by

$$\varphi_{\iota,\varrho} = \varphi_\varrho \circ \mu\ , \tag{4.7}$$

where the unique conditional expectation μ of B onto A coincides with the left-inverse ϕ_ι. Since μ maps $(\iota\circ\varrho^n, \iota\circ\varrho^m)$ onto (ϱ^n, ϱ^m), it maps $\mathcal{O}_{\iota,\varrho}$ onto $\mathcal{O}_\varrho$. Similarly as before, one has

$$\varphi_{\iota,\varrho}(VV^*) = \frac{d(\sigma)}{d(\iota)d(\varrho)^n} \tag{4.8}$$

whenever $V \in (\sigma, \iota\circ\varrho^n)$ is an isometry and $\sigma \prec \iota\circ\varrho^n$ is an irreducible subsector of $\iota\circ\varrho^n$.

The faithful state $\varphi_{\iota,\varrho}$ gives rise, by the GNS construction, to the Hilbert space $\mathcal{H}_{\iota,\varrho}$ which is the closure of the pre Hilbert space $\mathcal{O}_{\iota,\varrho}$. Specializing to $B = A$, $\iota = \mathrm{id}_A$ as before, the state φ_ϱ gives rise to the GNS Hilbert space $\mathcal{H}_\varrho \supset \mathcal{O}_\varrho$.

Let us now return to our Example 3.4. We were decomposing an element of $\mathcal{O}_\varrho$ into a part generated by the skeleton space K_ϱ, and a remainder term $(U_c v_1^{R+1})(U_d v_1^{R+1})^*$. This term converges to zero when considered as an element of the GNS Hilbert space $\mathcal{H}_\varrho$, namely

$$\begin{aligned} \|(U_c v_1^{R+1})(U_d v_1^{R+1})^*\|_{\varphi_\varrho}^2 &= \varphi_\varrho[(U_d v_1^{R+1})(U_c v_1^{R+1})^*(U_c v_1^{R+1})(U_d v_1^{R+1})^*] \\ &= \varphi_\varrho[(U_d v_1^{R+1})(U_d v_1^{R+1})^*] \\ &= d(\varrho)^{-n-R-1}\ . \end{aligned} \tag{4.9}$$

by (4.6). It is, however, crucial that this is no convergence in the C*-algebra $\mathcal{O}_\varrho$, since every remainder term is a partial isometry, and hence has unit norm.

The example already illustrates the general scheme. In fact, we have

4.1 Lemma. *Let $E_n = \sum_\alpha v_\alpha v_\alpha^*$ be the support projection of the Hilbert space k_n where $\{v_\alpha\}$ is an orthonormal basis of $k_n \subset K_{\iota,\varrho}$. Then the expectation value $\varphi_{\iota,\varrho}(\sum_n E_n)$ converges to 1.*

Proof. Since $v_i \in (\iota, \iota\circ\varrho^n)$, one has $\varphi_{\iota,\varrho}(v_i v_i^*) = d(\varrho)^{-n}$ due to (4.8), and $\varphi_{\iota,\varrho}(E_n) = \dim(k_n) d(\varrho)^{-n}$. It follows that the sum $\sum_n \varphi_{\iota,\varrho}(E_n)$ converges to the value of the generating functional $k(t)$ at $t = 1/d(\varrho)$ while to evaluate $k(t)$ at t below $1/d(\varrho)$ amounts to suppress the higher contributions to the sum. But since $h(t)$ diverges as $t \nearrow 1/d(\varrho)$, we conclude by Lemma 3.3 that $k(1/d(\varrho)) = 1$. □

4.2 Corollary. *$\sum_n E_n$ converges to $\mathbb{1}_B$ within the Hilbert space $\mathcal{H}_{\iota,\varrho}$. If $K_{\iota,\varrho}$ is finite-dimensional, then it has unit support, $\sum_n E_n = \mathbb{1}_B$.*

Proof. Obvious, since E_n are orthogonal projections by Lemma 3.1. □

4.3 Corollary. $\dim(H_{\iota,\varrho}) \le d(\varrho) \le \dim(K_{\iota,\varrho})$. *Equality in either of these bounds implies equality in the other one, and holds if and only if either of the following is true:*
(i) *$K_{\iota,\varrho} = k_1 \equiv H_{\iota,\varrho}$.*
(ii) *$H_{\iota,\varrho}$ has unit support.*
(iii) *ϱ is implemented by $K_{\iota,\varrho}$.*
(iv) *ϱ is implemented by $H_{\iota,\varrho}$.*

Proof. The first bound holds because $\dim(H_{\iota,\varrho})$ is the multiplicity of ι within $\iota\circ\varrho$, with equality if and only if (iv) is true. The second bound holds since every basis isometry $v_\alpha \in k_n$ contributes at most $1/d(\varrho)$ to the sum $\sum_{n,\alpha} \varphi_{\iota,\varrho}(v_\alpha v_\alpha^*)$ $= 1$, with equality if and only if each contribution is exactly $1/d(\varrho)$, which in turn is equivalent to (i).

Equivalence of (i)–(iv) is seen as follows: if $K_{\iota,\varrho} = H_{\iota,\varrho}$ then it is finite-dimensional, hence it has unit support, hence $K_{\iota,\varrho} = H_{\iota,\varrho}$ implements ϱ. Conversely, if $H_{\iota,\varrho}$ implements ϱ, then its support must be unity, and if $H_{\iota,\varrho}$ has unit support, then $K_{\iota,\varrho}$ cannot be larger than $H_{\iota,\varrho}$. If $K_{\iota,\varrho}$ implements ϱ, then $K_{\iota,\varrho} \subset (\iota, \iota\circ\varrho) = H_{\iota,\varrho}$. In either case, $K_{\iota,\varrho} = H_{\iota,\varrho}$ follows. □

In a similar way as Lemma 4.1, one obtains

4.4 Lemma. *Every element of $(\iota\circ\varrho^n, \iota\circ\varrho^m)$ can be approximated (in the Hilbert norm of $\mathcal{H}_{\iota,\varrho}$) by sums of operators in $h_{m+r} h_{n+r}^*$.*

Proof. Every $x \in (\iota\circ\varrho^n, \iota\circ\varrho^m)$ is a sum of operators $T_a T_b^*$ and $S_c S_d^*$ with isometries $T_a \in (\iota, \iota\circ\varrho^m), T_b \in (\iota, \iota\circ\varrho^n)$ and $S_c \in (\sigma_j, \iota\circ\varrho^m), S_d \in (\sigma_j, \iota\circ\varrho^n)$ where σ_j ranges over all irreducible subsectors contained in $\iota\circ\varrho^n$, $n \in \mathbb{N}$, which are different from ι. The terms $T_a T_b^*$ are of the desired form. The latter are treated as follows. $(\sigma_j, \iota\circ\varrho^n)$ is a subspace of $(\sigma_j\circ\varrho, \iota\circ\varrho^{n+1})$. According to the fusion rules $\sigma_j\circ\varrho \cong (N_{\iota,\varrho})_{\iota j}\iota \oplus \sum_{i\neq\iota}(N_{\iota,\varrho})_{ij}\sigma_i$ we may rewrite

$$S_c S_d^* = \sum_\alpha S_c w_\alpha w_\alpha^* S_d^* + \sum_{i,\alpha} S_c v_{i,\alpha} v_{i,\alpha}^* S_d^* \tag{4.10}$$

where $\{w_\alpha\}$ is an orthonormal basis of $(\iota, \sigma_j\circ\varrho)$ and $\{v_{i,\alpha}\}$ are orthonormal bases of $(\sigma_i, \sigma_j\circ\varrho)$. The former terms are in $h_{m+1} h_{n+1}^*$, while the latter are

of the form $\tilde{S}_c\tilde{S}_d^*$ with isometries $\tilde{S}_c \in (\sigma_i, \iota\circ\varrho^{m+1}), \tilde{S}_d \in (\sigma_i, \iota\circ\varrho^{n+1})$ $(i \neq \iota)$. Repeating this step R times, we obtain a sum of terms in $h_{m+r}h_{n+r}^*$ $(r \leq R)$ and a remainder r_R of the form

$$r_R = S_c \left(\sum v_1 \dots v_R v_R^* \dots v_1^* \right) S_d^* \tag{4.11}$$

where $v_r = v_{i_r,\alpha_r}$ and the sum extends over all $i_r \neq \iota$ and over orthonormal bases $\{v_{i_r,\alpha_r}\}$ of $(\sigma_{i_r}, \sigma_{i_{r-1}}\circ\varrho)$. The Hilbert norm of the remainder is, by (4.8),

$$\|r_R\|^2_{\varphi_{\iota,\varrho}} = \varphi_{\iota,\varrho}(r_R^* r_R) = \sum \varphi_{\iota,\varrho}\left[S_d v_1 \dots v_R v_R^* \dots v_1^* S_d^*\right] = \sum \frac{d_{i_R}}{d(\varrho)^{n+R}} \,. \tag{4.12}$$

The number of terms in this sum with fixed sector $i_R = i$ equals $(N_{\rm red}^R)_{ij}$ where $N_{\rm red} = QN_{\iota,\varrho}Q$ is the fusion matrix with the line and column corresponding to ι deleted. This yields the estimate

$$\|r_R\|^2_{\varphi_{\iota,\varrho}} \leq const. \cdot \left(\frac{\|N_{\rm red}\|}{d(\varrho)} \right)^R \,. \tag{4.13}$$

The norm of the reduced fusion matrix is strictly smaller than the norm $\|N_{\iota,\varrho}\| = d(\varrho)$ of the full fusion matrix, because the Frobenius eigenvector has a nonzero component in the ι-direction. It follows that (4.13) converges to zero as R increases. □

4.5 Corollary. *The following statements are equivalent.*
(i) $\mathcal{O}_{\iota,\varrho}$ *is generated by its skeleton space* $K_{\iota,\varrho}$, *that is* $\mathcal{O}_{\iota,\varrho} = \mathcal{O}(K_{\iota,\varrho})$.
(ii) $K_{\iota,\varrho}$ *is finite-dimensional.*
(iii) *The "reduced fusion matrix"* $N_{\rm red} = QN_{\iota,\varrho}Q$ (*cf. Eq.* (3.3)) *is nilpotent.*

Proof. For the equivalence (i) ⇔ (iii), consider the expansion discussed in the proof to Lemma 4.4. Since every term in the remainder r_R is a partial isometry of norm 1, the expansion converges within $\mathcal{O}_{\iota,\varrho}$, that is in norm, if and only if the remainder vanishes for sufficiently large R. This is equivalent to nilpotency of $N_{\rm red}$. The equivalence (ii) ⇔ (iii) is obvious from Eq. (3.3). □

5. Discussion

We have introduced a new class of generalized Cuntz algebras $\mathcal{O}_{\iota,\varrho}$ motivated by the analysis of charged operators, that is, operators in an ambient algebra B which intertwine a given endomorphism ϱ of a subalgebra A with the identity id_A.

We have studied the embedding of "higher" charged operators, that is, intertwiners in B between the identity and powers of $\varrho \in End(A)$, into the associated generalized Cuntz algebra. An interesting invariant object turns out to be the "skeleton space" $K_{\iota,\varrho}$ within $\mathcal{O}_{\iota,\varrho}$. In the fixpoint example, the skeleton space of $\mathcal{O}_{\iota,\varrho}$ is just the Hilbert space $H_{\iota,\varrho}$ of charged operators which implements ϱ and carries a representation of the Hopf algebra G, while the skeleton space of $\mathcal{O}_\varrho$ collects a system of algebraically independent G-invariants within tensor powers of the representation on $H_{\iota,\varrho}$. In this sense, we are setting up a generalized theory of invariants.

The skeleton space gives rise to an embedding $\mathcal{O}(K_{\iota,\varrho}) \subset \mathcal{O}_{\iota,\varrho}$. The embedding of ordinary Cuntz algebras into generalized ones itself is not so much surprising in view of the elementary fact that ordinary Cuntz algebras generated by d isometries can be embedded into Cuntz algebras generated by less than d isometries (e.g., if $\mathcal{O}_2$ is generated by v_1 and v_2, then $v_1^r v_2$ $(r = 0, \ldots d-2)$ and v_1^{d-1} span a Hilbert space of dimension d with unit support within $\mathcal{O}_2$). But the simple criterion for equality of $\mathcal{O}_{\iota,\varrho}$ with $\mathcal{O}(K_{\iota,\varrho})$ in terms of the fusion rules (Corollary 4.5) seems interesting, possibly leading to a classification of these exceptional cases. The (rather weak) convergence of the support of the Hilbert space $K_{\iota,\varrho}$ in the general case may be irrelevant for the properly C* aspects, but in view of the motivating problem in which the states $\varphi_{\iota,\varrho}$ and φ_ϱ are natural tools, we expect it to be of interest for the study of generalized symmetry.

The criterion whether $\mathcal{O}_{\iota,\varrho}$ is an ordinary Cuntz algebra distinguishes only the case when the generating Hilbert space coincides with the skeleton space $K_{\iota,\varrho}$. The Example 3.4 shows that possibly $\mathcal{O}_{\iota,\varrho} = \mathcal{O}(H)$ with some $H \neq K_{\iota,\varrho}$. In that example, $K_{\iota,\varrho}$ is infinite-dimensional while $\dim(H) = 2$. The skeleton space therefore does not provide a characterization of the full isomorphism class of the new generalized Cuntz algebras.

However, the skeleton spaces and the generating functionals $k(t)$, as well as the relative position of $\iota(\mathcal{O}_\varrho)$ in $\mathcal{O}_{\iota,\varrho}$ are invariants under inner conjugations and outer transformations $(A, B, \iota, \varrho) \mapsto (\hat{A}, \hat{B}, \beta\circ\iota\circ\alpha^{-1}, \alpha\circ\varrho\circ\alpha^{-1})$ with $\alpha \in Iso(A, \hat{A})$ and $\beta \in Iso(B, \hat{B})$. These data, as ϱ ranges over the even vertices of the principal graph determined by ι, therefore constitute invariant information about the underlying paragroup.

References

[1] T. Ceccherini, S. Doplicher, C. Pinzari, J.E. Roberts: *A generalization of the Cuntz algebras and model actions*, J. Funct. Anal. **125**, 416–437 (1994).

[2] J. Cuntz: *Simple C*-algebras generated by isometries*, Commun. Math. Phys. **57**, 173–185 (1977).

[3] S. Doplicher, J.E. Roberts: *A new duality theory for compact groups*, Invent. Math. **98**, 157–218 (1989).

[4] S. Doplicher, J.E. Roberts: *Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics*, Commun. Math. Phys. **131**, 51–107 (1990).

[5] M. Izumi: *Subalgebras of infinite C*-algebras with finite Watatani indices. I. Cuntz algebras*, Commun. Math. Phys. **155**, 157–182 (1993).

[6] R. Longo: *Index of subfactors and statistics of quantum fields. I*, Commun. Math. Phys. **126**, 217–247 (1989).

[7] R. Longo: *A duality for Hopf algebras and for subfactors. I*, Commun. Math. Phys. **159**, 133–150 (1994).

[8] R. Longo, K.-H. Rehren: *Nets of subfactors*, Rev. Math. Phys. **7**, 567–597 (1995).

[9] M. Pimsner: *A class of C*-algebras generalizing both Cuntz-Krieger algebras and crossed products by* $\mathbb{Z}$, Univ. of Pennsylvania preprint (1996).