SUPERSELECTION SECTORS WITH BRAID GROUP STATISTICS AND EXCHANGE ALGEBRAS II: GEOMETRIC ASPECTS AND CONFORMAL COVARIANCE

Dedicated to Rudolf Haag on the occasion of his 70th birthday

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The general theory of superselection sectors is shown to provide almost all the structure observed in two-dimensional conformal field theories. Its application to two-dimensional conformally covariant and three-dimensional Poincaré covariant theories yields a general spin-statistics connection previously encountered in more special situations. CPT symmetry can be shown also in the absence of local (anti-) commutation relations, if the braid group statistics is expressed in the form of an exchange algebra.

1. Introduction

Conformal field theories in two dimensions exhibit a rich spectrum of superselection sectors which gives rise to braid group representations, link invariants, and fusion categories of quantum symmetries. Actually, most of this structure does not really depend on conformal covariance but only on locality, and occurs in generic quantum field theories in two and three dimensions. We dedicate this article to Rudolf Haag on the occasion of his 70th birthday, who always emphasized the central role of the principle of Locality in quantum field theory [1, 2].

In [3] we started a general analysis of quantum field theories in low-dimensional space-time by adapting the Doplicher-Haag-Roberts (DHR) theory [4] of superselection sectors and statistics. For further aspects of this approach we recommend Rudolf Haag's recent book [2]. For related work in low dimensions see Buchholz *et al.* [5], Longo [6], and Fröhlich *et al.* [7], and contributions of the same authors to [8], where also reviews by Roberts and by Kastler *et al.* can be found.

The new feature of the DHR theory in low-dimensional space-time is braid group statistics: there are two distinct statistics operators, one the inverse of the other,

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related to the fact that the causal complement of a bounded space-time region has two connected components. The statistics operator as defined in [4] is a topological invariant w.r.t. the choice of a pair of causally disjoint auxiliary regions, as long as such pairs can be continuously deformed into each other while remaining at relative space-like separation. Therefore, braid group statistics arises for DHR superselection charges localized in two-dimensional double cones (diamonds) as well as in one-dimensional intervals of the real line (light-cone). It also arises in three-dimensional theories with charges localized along narrow cones extending to space-like infinity ("gauge strings") since in this situation the cones of localization are restricted to avoid the causal shadow of some space-like reference direction [9]. This type of localization, fitting the heuristic Mandelstam formula for charge-carrying fields in gauge theories, has been shown to be the most general situation with massive particles in at least three dimensions [9]. The most general particle sectors in two-dimensional massive theories, however, are solitons for which a general theory of sectors is not yet available (in typical examples, the structure has been investigated by Fröhlich [10], and a proposal for a general theory has been made in [11]).

These various algebraic localization concepts pertaining to relativistic quantum field theories have their (less precisely defined) counterparts in localized states of condensed matter physics. It is generally expected that they help to explain a class of "new" low-dimensional phenomena including the fractional Hall effect.

In [3] we showed how unitary braid group representations and Markov traces arise from the DHR theory, thereby reproducing the Jones-Ocneanu-Wenzl representations and the associated link invariants. We introduced a general formalism to describe charged operators which interpolate among different superselection sectors: the "reduced field bundle" $\mathcal{F}_{\text{red}}$ which is obtained from the "field bundle" (the crossed product of the algebra of observables by the semi-group of DHR endomorphisms [4]) by eliminating the redundancy. It was identified with an abstract version of the "exchange algebra" which has been found by two of us [12] as the algebraic structure underlying the conformal blocks [13] and their monodromy and fusion behaviour [14, 15].

The reduced field bundle $\mathcal{F}_{\text{red}}$ (see Sec. 3) is an algebra densely spanned by operators $F = F(e, A)$, linear in the local degree of freedom $A \in \mathcal{A}$ (the algebra of observables), and with a multi-index e referring to the charge $c(e)$ carried by F as well as to the source sector $s(e)$ and the range sector $r(e)$ between which F interpolates according to the "fusion rules". To be precise, for every superselection sector (equivalence class of irreducible DHR representations) $[\alpha]$ with finite statistics, one picks a representative DHR endomorphism ρ_α inducing a representation of the observable algebra $\mathcal{A}$ on a copy $\mathcal{H}_\alpha \equiv (\alpha, \mathcal{H}_0)$ of the vacuum Hilbert space by

$$\pi_\alpha(A) \cdot (\alpha, \Psi) := (\alpha, \pi_0(\rho_\alpha(A)) \cdot \Psi) . \tag{1.1}$$

For a given representative ρ and every pair of representatives ρ_α, ρ_β such that $[\rho_\beta]$ arises as a subrepresentation of $[\rho_\alpha \rho]$, there is an orthonormal basis $\{T_e\}$ of the corresponding finite-dimensional space of local intertwiners : $\rho_\beta \to \rho_\alpha \rho$. The operators $F(e, A)$ carrying charge $[\rho]$ are defined by their action on $\mathcal{H}_\alpha$ (the source sector) with images in $\mathcal{H}_\beta$ (the range sector):

$$F(e, A) \cdot (\alpha, \Psi) := (\beta, \pi_0(T_e^* \rho_\alpha(A)) \cdot \Psi) . \tag{1.2}$$

Operators of trivial charge ($c(e) = [0]$, $T_e = 1$) coincide with the observables (1.1). $F = F(e, A)$ is said to be localized in some space-time region $\mathcal{O}$ (i.e. $F \in \mathcal{F}_{\text{red}}(\mathcal{O})$) iff it commutes with all observables localized in the causal complement of $\mathcal{O}$:

$$\pi_\beta(B) F(e, A) = F(e, A) \pi_\alpha(B) \quad \forall\, B \in \mathcal{A}(\mathcal{O}') . \tag{1.3}$$

This condition is equivalent to the existence of a local unitary U such that

$$U\, A \in \mathcal{A}(\mathcal{O}) \quad \text{and} \quad Ad_U \circ \rho \text{ is localized in } \mathcal{O} . \tag{1.4}$$

The notion of localization does not depend on the source and range sectors.

The algebraic relations satisfied by these reduced field bundle elements are exactly the bounded operator analogue of the exchange algebra introduced in [12] in the context of conformal quantum field theory on the light-cone, and it is the first intention of this article to deeper establish this correspondence. In particular, we shall prove the Spin-Statistics theorem, relating the statistics phase κ of the DHR theory to the appropriate spin quantum number of the outer (space-time) symmetry group (the light-cone scaling dimension in conformal quantum field theory), and derive the CPT theorem for *charged fields*, connecting DHR charge conjugation with space-time inversion.

Therefore, we shall concentrate on sectors covariant w.r.t. appropriate (Poincaré, conformal) space-time symmetry groups. In a recent article [16] Borchers showed that in the vacuum sector the Tomita-Takesaki modular theory associates with a positive-energy representation of the translation group a representation of the light-cone dilatations resp. of the Poincaré group in two dimensions, as well as an abstract CPT operator, although the geometric actions of these groups on the local net cannot always be guaranteed. As a partial converse of the result [17] that covariant sectors with an isolated mass-shell have finite statistics, Guido and Longo [18] have established that — under some regularity assumptions about the local net — a DHR sector with finite statistics is Poincaré covariant, provided the vacuum sector is. The same statement holds as well for Möbius invariance on the compactified chiral light-cone. In our presentation below we shall treat the selection criterion of finite statistics independently of the covariance assumption. The results of Sec. 3 rely only on finite statistics, while specific covariance assumptions about the vacuum sector and the charged sectors enter only in the later sections.

In Sec. 2 we shall briefly review the results of the DHR theory of superselection sectors with braid group statistics. In Sec. 3 we shall derive consequences of the DHR theory for the algebraic structure of the reduced field bundle, in particular as the operator adjoint and the charge conjugation operation are concerned. These results appear as a "pre-existing covariance" in the general DHR theory, presumably related to the results of Borchers [16] and Guido and Longo [18]. In Sec. 4

we review the basic properties of the space-time covariant reduced field bundle. Up to this point, space-time is assumed to be a non-compact flat manifold with a definite notion of left and right causal complement. In Sec. 5 we turn to the theory in non-trivial topology, in particular the compactified light-cone S^1 which supports the chiral observables in two-dimensional conformal quantum field theory. In such situations, the algebra of local observables must be extended to include certain global operators. We construct this "universal algebra" $\mathcal{A}_{\rm univ}$ and derive the existence of DHR endomorphisms of $\mathcal{A}_{\rm univ}$. The center of the universal algebra contains "charge operators" which in typical cases completely characterize the sector in which they are evaluated. The underlying algebra is an abstract Verlinde algebra [19], and the corresponding "character table" is Verlinde's matrix S [20, 7c]. The vacuum representation π_0 of the universal algebra is no longer faithful, and care has to be taken about pre-images of π_0. As a consequence, the reduced field bundle is found to "live" on a covering space with specific periodicity properties. The latter are specified by both the (conformal) spin and the statistics monodromy: their coincidence gives rise to a weak Spin-Statistics theorem, which in turn allows to interprete the charge quantum numbers showing up in the structure of the center of $\mathcal{A}_{\rm univ}$ in terms of the spin quantum numbers associated with the covariant sectors. These results hold on the conformal light-cone as well as in 2 + 1-dimensional theories with string-like charges. For further analysis including the strong Spin-Statistics theorem we restrict to the conformal situation, although we think that generalizations are possible. Provided the scaling limit is sufficiently well-behaved, we can control complex scale transformations $x \mapsto e^{i\varphi}x$ on the original light-cone, and in particular the space-time inversion $x \mapsto -x$. This, together with the algebraic charge conjugation structure of the reduced field bundle, leads us to the CPT theorem for charged sectors with braid group statistics.

Some of these results have been announced earlier (e.g. [20, 21]). The (weak version of the) Spin-Statistics theorem was also found by Fröhlich *et al.* [7] in the three-dimensional context. Our main interest lies in the study of ***non-abelian*** ("plektonic") braid group statistics going along with branching in the composition of sectors. The much simpler abelian ("anyonic") case which seems to be the only one accessible by Lagrangian methods is, however, always included.

We include two appendices, one containing a collection of useful formulae relevant for the charge conjugation structure of the reduced field bundle, the other one describing the construction of topological invariants from the superselection structure data of a local quantum field theory.

The condensed matter counterpart of relativistic space-time characteristics such as the spin are the critical exponents. Under this aspect, our above-mentioned findings about the center of the universal algebra of observables should be seen in the light of Kadanoff's *et al.* [22] understanding (by the Coulomb gas method) of critical exponents as being related to field theoretical charge quantum numbers. Charge quantum numbers related to the covering of the Möbius group have also been observed before and exploited for a conformal decomposition theory in pure quantum field theoretical studies [23]. But a rich class of models illustrating the non-triviality of these early concepts and triggering the modern developments in low-dimensional quantum field theory was only found much later [13].

We do not address the question of "quantum symmetry" in the present article, i.e. the analogue of the result of Doplicher and Roberts that DHR superselection sectors with permutation group statistics are always due to a global gauge symmetry in a field algebra of bosonic or fermionic fields [24] (see, however, [25,26,27] for approaches in the algebraic framework). The reduced field bundle is a substitute for the gauge covariant field algebra when the symmetry is not known. Since the former is a faithful description of all physics observable in charged states, it is only natural that the deep physical properties of charged fields can be derived as well in our framework without explicit reference to gauge covariance.

Actually, it is known that the reduced field bundle framework for gauge covariant theories is essentially equivalent to the much older multiplicity-free description introduced by Drühl *et al.* [28], who succeeded to construct Green's parafields [29] (generalized permutation group statistics) from gauge tensor fields with the help of a Klein transformation.

2. Preliminaries from the DHR Theory

The general setting of the DHR theory is the *local net of observables* which assigns to every bounded space-time region $\mathcal{O}$ the von Neumann algebra $\mathcal{A}(\mathcal{O})$ of local observables localized in $\mathcal{O}$. $\mathcal{A}$ is the C^*-algebra generated by all $\mathcal{A}(\mathcal{O})$. The vacuum representation $\pi_0 : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H}_0)$ is faithful, and Haag duality is assumed for double-cones $\mathcal{O}$ (intersections of a forward and a backward light-cone), i.e. every operator in $\mathcal{B}(\mathcal{H}_0)$ which commutes with the algebra of the causal complement $\pi_0(\mathcal{A}(\mathcal{O}'))$ is the image of a local operator in $\mathcal{A}(\mathcal{O})$. A physically relevant class of Hilbert space positive energy representations is obtained by the composition $\pi_0 \circ \rho$ of the vacuum representation $\pi_0 : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H}_0)$ with an *endomorphism* $\rho : \mathcal{A} \longrightarrow \mathcal{A}$. The latter is localized and transportable, i.e. it acts trivially on the algebra of the causal complement $\mathcal{O}'$ of a given double-cone $\mathcal{O}$ (then ρ is said to be localized in $\mathcal{O}$), and there are unitarily equivalent endomorphisms localized in any other bounded region. The physically most relevant endomorphisms have "finite statistics". By Longo's result [6], this amounts to saying that the inclusion of factors $\rho(\mathcal{A}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O})$ has finite (Jones) index. The composition of DHR endomorphisms gives rise to a "fusion category" of superselection sectors. We recall some basic notions and results from [3, 4].

2.1. *Intertwiners*

Two representations $\pi_0\sigma$ and $\pi_0\rho$ possess equivalent subrepresentations iff there is an intertwining operator $\mathcal{V} : \pi_0\rho \longrightarrow \pi_0\sigma$ in $\mathcal{B}(\mathcal{H}_0)$, i.e. $\mathcal{V} \cdot \pi_0\rho(A) = \pi_0\sigma(A) \cdot \mathcal{V}$ for all $A \in \mathcal{A}$. The equivalent subrepresentations are given by the source and range projections of $\mathcal{V}$. By Haag duality, $\mathcal{V}$ is the image in π_0 of a *local* intertwiner $T : \rho \longrightarrow \sigma$ in $\mathcal{A}$ satisfying

$$T\rho(A) = \sigma(A)T \qquad \forall\, A \in \mathcal{A}\,. \tag{2.1}$$

The space of self-intertwiners: $\rho \longrightarrow \rho$ is the commutant $\rho(\mathcal{A})'$ of $\rho(\mathcal{A})$ and, by Schur's Lemma, equals the scalars $\mathbb{C}$ iff ρ is irreducible. Therefore, when ρ is irreducible, the linear space of intertwiners: $\rho \longrightarrow \sigma$ is a *Hilbert space within the algebra* of local observables with the inner product $(T_1, T_2) := T_1^* T_2 \in \mathbb{C}$.

2.2. *Statistics*

For every pair of DHR endomorphisms there is a unitary local intertwiner $\varepsilon(\rho, \sigma) : \rho\sigma \longrightarrow \sigma\rho$, the *statistics operator*. The collection of statistics operators is uniquely determined by the coherence with local intertwiners and among themselves

$$\varepsilon(\sigma_1, \sigma_2)\sigma_1(T_2)T_1 = T_2\rho_2(T_1)\,\varepsilon(\rho_1, \rho_2) \qquad \forall\, T_i : \rho_i \longrightarrow \sigma_i\,, \tag{2.2}$$

$$\varepsilon(\rho_1\rho_2, \sigma) = \varepsilon(\rho_1, \sigma)\rho_1(\varepsilon(\rho_2, \sigma)), \qquad \varepsilon(\rho, \sigma_1\sigma_2) = \sigma_1(\varepsilon(\rho, \sigma_2))\varepsilon(\rho, \sigma_1)\,, \tag{2.3}$$

together with the initial conditions

$$\varepsilon(\rho, \mathrm{id}) = \varepsilon(\mathrm{id}, \rho) = 1\,, \tag{2.4}$$

$$\varepsilon(\rho, \sigma) = 1 \qquad \text{whenever } \sigma < \rho\,, \tag{2.5}$$

where "$\sigma < \rho$" means that σ is localized in a region in the left space-like complement of another region where ρ is localized. (Note that id is localized in every region, and therefore (2.4) is actually contained in (2.5).) Imposing the opposite initial condition: trivialization for $\rho < \sigma$, would give rise to the opposite statistics operators $\varepsilon(\sigma, \rho)^*$ instead. In high-dimensional situations without an invariant distinction between left and right, permutation group statistics follows: $\varepsilon(\rho, \sigma) = \varepsilon(\sigma, \rho)^*$.

As a consequence of (2.2 + 3), the statistics operators satisfy the braid relation

$$\rho_3(\varepsilon(\rho_1, \rho_2))\varepsilon(\rho_1, \rho_3)\rho_1(\varepsilon(\rho_2, \rho_3)) = \varepsilon(\rho_2, \rho_3)\rho_2(\varepsilon(\rho_1, \rho_3))\varepsilon(\rho_1, \rho_2)\,. \tag{2.6}$$

In particular, by assigning the local operators $\rho^{i-1}(\varepsilon(\rho, \rho))$ to the standard generators σ_i of the braid group B_n, one obtains a unitary representation of the braid group in $\mathcal{A}$, called the statistics of the endomorphism ρ.

2.3. *Conjugates*

A *conjugate* of a DHR endomorphism ρ is a DHR endomorphism $\bar{\rho}$ such that $\bar{\rho}\rho$ contains the vacuum sector. More precisely, there is an isometric intertwiner $R : \mathrm{id} \longrightarrow \bar{\rho}\rho$ which induces a standard left-inverse ϕ of ρ

$$\phi(A) = R^*\bar{\rho}(A)R \tag{2.7}$$

with finite statistics. Recall that a left-inverse is a normalized positive linear map satisfying the relation $\phi(\rho(A)B\,\rho(C)) = A\phi(B)C$. It is called regular if it is of the form (2.7), and standard if in addition the statistics parameter $\lambda_\rho := \phi(\varepsilon(\rho,\rho)) \in \rho(\mathcal{A})'$ is a non-vanishing multiple of a unitary. A sufficient condition for the existence of a standard left-inverse and therefore of a conjugate is that there is *some* left-inverse with statistics parameter $\neq 0$ ("finite statistics") and ρ is translation covariant with the spectrum condition. The standard left-inverse is unique, and the conjugate endomorphism $\bar{\rho}$ is unique up to unitary equivalence and depends only on the equivalence class of ρ. ρ is conjugate to $\bar{\rho}$. The standard left-inverse is a trace on the commutant of $\rho(\mathcal{A})$, and $\phi_2\phi_1$ is the standard left-inverse of $\rho_1\rho_2$.

The inverse modulus of λ_ρ is called the statistical dimension $d_\rho \equiv d(\rho) \geq 1$ of ρ. For irreducible endomorphisms, the scalar $\lambda_\rho = \kappa_\rho/d_\rho$ defines the statistics phase $\kappa_\rho \equiv \kappa(\rho)$. Both d and κ are class invariant quantum numbers, and

$$d(\rho) = d(\bar{\rho}), \qquad \kappa(\rho) = \kappa(\bar{\rho})\,. \tag{2.8}$$

For reducible endomorphisms, the statistics parameter has the form

$$\lambda_\rho = d_\rho^{-1} \sum_\alpha \kappa_\alpha E_\alpha\,, \tag{2.9}$$

where $E_\alpha \in \rho(\mathcal{A})'$ are the projections onto the equivalence classes $[\alpha]$ of irreducible subrepresentations of $\pi_0 \circ \rho$.

The statistical dimension is the square root of the minimal index [6] of the inclusion $\rho(\mathcal{A}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O})$ which coincides with the Jones index in the irreducible case. It is multiplicative and additive:

$$\text{(a)}\ d(\rho_1\rho_2) = d(\rho_1)d(\rho_2) \qquad \text{(b)}\ d(\rho) = \sum_\alpha N^\alpha d(\rho_\alpha)\,, \tag{2.10}$$

where N^α are the multiplicities of the sectors $[\alpha]$ within $\pi_0 \circ \rho$. As a consequence, every product of irreducible endomorphisms from the sectors $[\alpha]$, $[\beta]$ with finite statistics is finitely reducible into sectors $[\gamma]$ with finite statistics with multiplicities

$$N^\gamma_{\alpha\beta} = N^\gamma_{\beta\alpha} = N^\alpha_{\gamma\bar{\beta}} = N^{\bar{\alpha}}_{\bar{\gamma}\beta} = N^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}}\,, \quad N^0_{\bar{\alpha}\beta} = N^\beta_{0\alpha} = \delta_{\alpha\beta}\,. \tag{2.11}$$

3. The Reduced Field Bundle

We shall from now on fix a set Δ_{red} of irreducible representative endomorphisms with finite statistics $\rho_\alpha \in [\alpha]$, one per superselection sector, with id $\in [0]$ as representative of the vacuum sector. We also fix orthonormal bases $\{T_e\}$ for all the $N^\beta_{\rho\alpha}$-dimensional intertwiner spaces: $\rho_\beta \longrightarrow \rho_\alpha\rho$. We call $[\alpha] = s(e)$, $[\rho] = c(e)$, $[\beta] = r(e)$ the source, charge, and range of the corresponding fusion channel, and say e to be of type (α, ρ, β). Although, as a rule, we shall distinguish by our notation the physical role of the source and range sectors and their representatives ($\rho_\alpha \in [\alpha]$, $\rho_{\bar{\alpha}} \in [\bar{\alpha}]$ etc.) from charge quantum numbers ($\rho_i \in [\rho_i]$, $\bar{\rho} \in [\bar{\rho}]$), the endomorphisms always are taken from the same set Δ_{red}. We have the orthonormality and completeness relations

$$T_e^* T_f = \delta_{ef}\,, \quad \sum_e T_e T_e^* = 1\,, \tag{3.1}$$

for e and f with common source and charge, but free range (to be summed over in the completeness relation). We choose $T_e = 1$ if e has trivial charge or source, and call $T_e = R_\rho = R_{\bar{\alpha}}$ if the range is trivial, i.e. charge and source are conjugates.

For later use we introduce the complex phases associated with $\rho \in \Delta_{\rm red}$:

$$\chi_\rho := d_\rho \cdot \bar\rho(R^*_{\bar\rho})R_\rho \equiv d_\rho \cdot R^*_{\bar\rho}\rho(R_\rho) \equiv \kappa_\rho \cdot R^*_{\bar\rho}\varepsilon(\bar\rho,\, \rho)R_\rho \quad \Rightarrow \quad \chi_{\bar\rho} = \chi^*_\rho\,. \tag{3.2}$$

The phases χ are to some extent invariants of the sector, but they also depend on the phases of the intertwiners R_ρ relative to $R_{\bar\rho}$. For ρ not self-conjugate, these may be chosen such that $\chi_\rho = 1$. For self-conjugate sectors, χ_ρ is an *intrinsic* sign distinguishing "real" (+) from "pseudoreal" (−) sectors.

We denote by $\mathcal{H}_{\rm red}$ the direct sum of Hilbert spaces $\mathcal{H}_\alpha$, $\rho_\alpha \in \Delta_{\rm red}$. The reduced field bundle $\mathcal{F}_{\rm red} \subset B(\mathcal{H}_{\rm red})$ is now defined by (1.2). The reader should be aware that, from the definitions (1.1–3), the choice of the representatives $\rho_\alpha \in [\alpha]$ is completely immaterial up to unitary equivalence. It amounts to a global section through the field bundle defined in [4]. In fact, choosing $\rho'_\alpha = {\rm Ad}_{V_\alpha} \circ \rho_\alpha$, one naturally identifies $\mathcal{H}_\alpha$ with $\mathcal{H}'_\alpha$ by $\pi_0(V_\alpha) : \mathcal{H}_\alpha \to \mathcal{H}'_\alpha$ and replaces the basis T_e by $T'_e = V_\alpha\rho_\alpha(V_\rho)T_eV^*_\beta$. Then $F'(e,\, V_\rho A)$ is unitarily equivalent to $F(e,\, A)$. For fixed choice $\Delta_{\rm red}$, $F(e,\, A)$ transform contravariantly under a change of the basis $\{T_e\}$.

Before we present the algebraic properties of $\mathcal{F}_{\rm red}$, let us comment on the underlying "path space" structure. It is clear from the definition (1.2) that elements $F(e_2,\, A_2)$ and $F(e_1,\, A_1)$ of the reduced field bundle can be multiplied only if $s(e_2) = r(e_1)$; otherwise the product is set to zero. Therefore, a multiple product $F(e_n,\, A_n)\ldots F(e_1,\, A_1)$ corresponds to a chain of fusion channels[2] $\eta = e_n \circ \ldots \circ e_1$ such that $s(e_{i+1}) = r(e_i)$ but with free charges $c(e_i)$ (those of the field bundle operators). We refer to such a chain as a "path". The repeated evaluation of (1.2) gives rise to "path intertwiners" $T_\eta = T_{e_1}\ldots T_{e_n} : \rho_{\beta_n} \to \rho_{\alpha_1}\rho_1\ldots\rho_n$ where $\rho_i \in c(e_i)$ etc. as in Sec. 1. Since T_e provide orthonormal bases of the intertwiner spaces: $\rho_\beta \to \rho_\alpha\rho$, T_η as above with α_1, ρ_i, β_n fixed but $\alpha_{i+1} = \beta_i$ free, provide orthonormal bases for the intertwiner spaces: $\rho_{\beta_n} \to \rho_{\alpha_1}\rho_1\ldots\rho_n$. Moreover, the "string intertwiners" $T_\eta T^*_\xi := T_{e_1}\ldots T_{e_n}T^*_{f_m}\ldots T^*_{f_1}$ with $s(e_1) = [\alpha]$, $c(e_i) = [\rho_i]$, $c(f_j) = [\rho'_i]$, $s(f_1) = [\gamma]$ fixed but all other sectors including $r(e_n) = r(f_m)$ free, provide a basis of the intertwiner spaces: $\rho_\gamma\rho'_1\ldots\rho'_m \to \rho_\alpha\rho_1\ldots\rho_n$. Therefore, e.g., for any intertwiner $T_f : \rho \to \rho_1\rho_2$ and statistics operator $\varepsilon(\rho_2,\, \rho_1) : \rho_2\rho_1 \to \rho_1\rho_2$ there are unique expansions

$$\begin{aligned} \rho_\alpha(T_f) &= \sum_{e_i,\, e} D^{e_2\circ e_1}_{f,e} \cdot T_{e_1}T_{e_2}T^*_e\,, \\ \rho_\alpha(\varepsilon(\rho_2,\, \rho_1)) &= \sum_{e_i, f_i} R^{e_2\circ e_1}_{f_1\circ f_2}(+) \cdot T_{e_1}T_{e_2}T^*_{f_1}T^*_{f_2} \end{aligned} \tag{3.3}$$

with unitary matrices D and $R(+)$ in path space. The corresponding matrix for the opposite statistics operator $\varepsilon(\rho_1,\, \rho_2)^*$ is called $R^{e_2\circ e_1}_{f_1\circ f_2}(-)$. The coefficients D and R play exactly the role of recoupling (Wigner-Racah, $6j$) symbols for the associativity resp. crossing symmetry of tensor products of representations of groups (Hopf algebras): e.g., $T_{e_1}T_{e_2}$ above corresponds to the successive reduction $(\rho_\alpha\rho_1)\rho_2 \to$

[2] The present notation $e_2 \circ e_1$, opposite to [3], conforms with the composition of arrows and maps, and reflects the interpolation of reduced field bundle operators.

$\rho_\beta\rho_2 \to \rho_\gamma$, while $\rho_\alpha(T_f)T_e$ corresponds to $\rho_\alpha(\rho_1\rho_2) \to \rho_\alpha\rho \to \rho_\gamma$. The transition matrix between these bases of intertwiners is the matrix D. Similarly, the R-matrices relate the successive reductions $\rho_\alpha\rho_1\rho_2 \to \rho_\beta\rho_2 \to \rho_\gamma$ and $\rho_\alpha\rho_2\rho_1 \to \rho_\delta\rho_1 \to \rho_\gamma$, where $\rho_1\rho_2$ and $\rho_2\rho_1$ are intertwined by the statistics operator. Observe that in both cases there are six sectors involved ("$6j$"). In fact, the role of these coefficients in the reduced field bundle (see below) is precisely that of $6j$-symbols of an underlying quantum symmetry: for sectors arising from a global gauge symmetry, this role can be easily established by the results of [24].

In view of these remarks, it is clear that also compositions of intertwiners such as those occur in (2.2), (2.3), and (2.6) have expansions in terms of path and string intertwiners. The composition of intertwiners corresponds to the matrix product in path space. In particular, the mentioned algebraic relations give rise to matrix identities among the structure constants R, D as follows. In (2.2) choose $\rho_1 = \sigma_1$ and $\rho_2 = \rho_\gamma$ from Δ_{red}, $\sigma_2 = \rho_\alpha\rho_\beta$ a product of two endomorphisms of Δ_{red}, $T_1 = 1$ trivial and $T_2 = T_e$ some basis intertwiner: $\rho_\gamma \to \rho_\alpha\rho_\beta$. Use (2.3) to express the statistics operator with the product $\rho_\alpha\rho_\beta$ in terms of statistics operators of the factors in Δ_{red}. This yields

$$\rho_\alpha(\varepsilon(\rho_1, \rho_\beta)) \cdot \varepsilon(\rho_1, \rho_\alpha) \cdot \rho_1(T_e) = T_e \cdot \varepsilon(\rho_1, \rho_\gamma) .$$

Apply some $\rho \in \Delta_{\text{red}}$ to this, and write $\rho\sigma(X) = \Sigma_f T_f \rho_\delta(X) T_f^*$ in the resulting equation, such that it becomes a relation among intertwiners of the type (3.3). Inserting their expansions (3.3) yields a matrix relation of the form $RRD = DR$, the Moore-Seiberg "pentagon" identity known from conformal quantum field theory [15]. The same manipulations on the braid relation (2.6) with all ρ_i from Δ_{red} yield a matrix relation of the form $RRR = RRR$, the braid relation on path space [12, 14]. Playing the same game on the expansion (3.3) $\rho(T)T = \Sigma D{\cdot}TT$ yields an identity of the form $DD = DDD$, known as Racah-Elliot relation in the context of (quantum) groups and tensor categories. It is often very useful and suggestive to display the algebraic intertwiner relations diagrammatically [25, 7], similar to the representation of the polynomial identities in [15], and to the diagrammatical tensor calculus of [30].

The structure constants R and D play a prominent role in the reduced field bundle. For the operators defined by (1.2) with the notion of localization (1.4) one easily derives

Proposition 3.1. *With notations as in (3.3):*

$$F(e_2, A_2) \cdot F(e_1, A_1) = \sum_{f,e} D_{f,e}^{e_2 \circ e_1} \cdot F(e, T_f^* \rho_1(A_2) A_1) , \tag{3.4}$$

and whenever F_2 is localized in the right/left causal complement of F_1,

$$F(e_2, A_2) \cdot F(e_1, A_1) = \sum_{f_1 \circ f_2} R_{f_1 \circ f_2}^{e_2 \circ e_1}(+/-) \cdot F(f_1, A_1) \cdot F(f_2, A_2) . \tag{3.5}$$

These are the characteristic exchange algebra relations with expansion coefficients indexed by appropriate fusion paths, replacing the labelling by representations and representation indices of conventional, say gauge covariant, fields. In (3.4) the charges contributing to the r.h.s. are precisely those arising as subsectors of $[\rho_1\rho_2]$ (i.e. the reduced field bundle respects the fusion rules), and in (3.5) the charges of the involved operators are just exchanged, as indicated by the subscripts $i = 1, 2$. The structure (3.4), as an abstract version of the "reduced matrix elements" of tensor operators in the Wigner-Eckart theorem of quantum mechanics, hints at the (in the case of braid group statistics unknown) underlying quantum symmetry.

Apart from the polynomial identities mentioned above, there is an inversion formula for the R-matrices.

Proposition 3.2. *Let $s(e_1) = s(f_2) = \alpha$, $r(e_2) = r(f_1) = \gamma$, $s(e_2) = r(e_1) = \beta$, $s(f_1) = r(f_2) = \delta$. Then (cf. also [7])*

$$R^{e_2\circ e_1}_{f_1\circ f_2}(-) = \frac{\kappa_\beta\kappa_\delta}{\kappa_\alpha\kappa_\gamma}\cdot R^{e_2\circ e_1}_{f_1\circ f_2}(+)\,. \tag{3.6}$$

This follows from the following lemma which will also be of relevance later for a weak version of the Spin-Statistics theorem. The same inversion formula was found in conformal exchange algebras as a consistency condition w.r.t. full Möbius covariance [12]. Its validity in the general DHR framework is another instance of pre-existing space-time covariance structures in the DHR theory.

Lemma 3.3. *The basis intertwiners $T_e : \rho_\gamma \to \rho_\alpha\rho_\beta$ diagonalize the monodromy (cf. also [7]):*

$$\varepsilon(\rho_\beta,\rho_\alpha)\varepsilon(\rho_\alpha,\rho_\beta)T_e = \frac{\kappa_\gamma}{\kappa_\alpha\kappa_\beta}\cdot T_e\,. \tag{3.7}$$

Basis intertwiners of common source and range are orthogonal also with respect to the left-inverse of the source:

$$\phi_\alpha(T_eT_f^*) = \frac{d_\gamma}{d_\alpha d_\beta}\,\delta_{ef}\,. \tag{3.8}$$

There are generalizations of the lemma: string intertwiners diagonalize the monodromy operators of n endomorphisms (the representatives of the braid generating the center of B_n), and are orthogonal w.r.t. the iterated left-inverses. To prove the lemma, one computes

$$\begin{aligned}
\lambda_\gamma\cdot\phi_\alpha(T_eT_f^*) &\equiv \lambda_\gamma\cdot\phi_\beta\phi_\alpha(T_eT_f^*) = \phi_\beta\phi_\alpha[T_e\rho_\gamma(R_{\bar\gamma}^*)\varepsilon(\rho_\gamma,\rho_\gamma)\rho_\gamma(R_{\bar\gamma})T_f^*]\\
&= \phi_\beta\phi_\alpha[\rho_\alpha\rho_\beta(R_{\bar\gamma}^*T_f^*)\varepsilon(\rho_\alpha\rho_\beta,\rho_\alpha\rho_\beta)\rho_\alpha\rho_\beta(T_eR_{\bar\gamma})]\\
&= R_{\bar\gamma}^*T_f^*\cdot\phi_\beta\phi_\alpha[\rho_\alpha[\varepsilon(\rho_\alpha,\rho_\beta)\rho_\alpha(\varepsilon(\rho_\beta,\rho_\beta))]\varepsilon(\rho_\alpha,\rho_\alpha)\rho_\alpha(\varepsilon(\rho_\beta,\rho_\alpha))]\\
&\quad\cdot T_eR_{\bar\gamma}\\
&= \lambda_\alpha\cdot R_{\bar\gamma}^*T_f^*\cdot\phi_\beta[\varepsilon(\rho_\alpha,\rho_\beta)\rho_\alpha(\varepsilon(\rho_\beta,\rho_\beta))\varepsilon(\rho_\beta,\rho_\alpha)]\cdot T_eR_{\bar\gamma}\\
&= \lambda_\alpha\lambda_\beta\cdot R_{\bar\gamma}^*\cdot T_f^*\varepsilon(\rho_\beta,\rho_\alpha)\varepsilon(\rho_\alpha,\rho_\beta)T_e\cdot R_{\bar\gamma}\\
&\equiv \lambda_\alpha\lambda_\beta\cdot T_f^*\varepsilon(\rho_\beta,\rho_\alpha)\varepsilon(\rho_\alpha,\rho_\beta)T_e\,,
\end{aligned}$$

where the identities $\equiv$ arise since $\phi(\cdot)$ resp. $R^* \cdot R$ are applied to scalars, the first equality follows from the definition of the statistics parameter λ_γ, the second equality is (2.2), the third equality uses (2.3) and the defining properties of left-inverses, the fourth equality makes use of the definition of λ_α and (2.7), and finally the last equality is the braid relation (2.6) together with the definition of λ_β. Now, without the λ-factors, the l.h.s. of the above equation is a positive hermitean matrix P, while the r.h.s. is a unitary matrix U in the indices e, f. Therefore, $P \cdot U^*$ must be the polar decomposition of $\lambda_\alpha \lambda_\beta / \lambda_\gamma \cdot \mathbb{1}$. This proves the lemma.

Proof of 3.2. We observe that by (2.2) and (2.3)

$$T^*_{e_1} \rho_\alpha(\varepsilon(\rho_2, \rho_1))\varepsilon(\rho_2, \rho_\alpha) = \varepsilon(\rho_2, \rho_\beta)\rho_2(T^*_{e_1})$$

and similarly

$$T^*_{e_1} \rho_\alpha(\varepsilon(\rho_1, \rho_2)^*)\varepsilon(\rho_\alpha, \rho_2)^* = \varepsilon(\rho_\beta, \rho_2)^* \rho_2(T^*_{e_1}) .$$

Elimination of $\rho_2(T^*_{e_1})$ yields

$$T^*_{e_1} \rho_\alpha(\varepsilon(\rho_1, \rho_2)^*) = \varepsilon(\rho_\beta, \rho_2)^* \varepsilon(\rho_2, \rho_\beta)^* \cdot T^*_{e_1} \rho_\alpha(\varepsilon(\rho_2, \rho_1)) \cdot \varepsilon(\rho_2, \rho_\alpha)\varepsilon(\rho_\alpha, \rho_2) .$$

Inserting this into the definition of the R-matrices and evaluating the monodromies by the lemma proves the claim.

In the following we shall establish *algebraic* structures of the reduced field bundle related to the *conjugation* structure of the superselection sectors. The latter is realized by several linear or anti-linear maps between charge-conjugation related intertwiner spaces of equal dimension (cf. (2.11)), e.g.,: $\rho_\beta \longrightarrow \rho_\alpha \rho$ and: $\bar\rho_\alpha \longrightarrow \bar\rho_\beta \rho$. These maps give rise to numerical matrices ("coupling constants") η, θ, ζ, whose definitions and relations are listed in Appendix A.

The first conjugation structure in the reduced field bundle is the ordinary adjoint of bounded operators in the Hilbert space $\mathcal{H}_{\rm red} = \oplus_\alpha \mathcal{H}_\alpha$. From the definition $\langle F^*\Phi, \Psi\rangle := \langle \Phi, F\Psi\rangle$ for $\Psi \in \mathcal{H}_\alpha$, $\Phi \in \mathcal{H}_\beta$ and (A.1), (3.2) one obtains

Proposition 3.4. *The reduced field bundle is closed under the* operator adjoint *of* $\mathcal{B}(\mathcal{H}_{\rm red})$ *which is*

$$F(e, A)^* = (d_\rho/\chi_\rho) \cdot \sum_{e^*} \eta_{ee^*} \, F(e^*, \bar\rho(A^*)R_\rho) . \tag{3.9}$$

Here, for e of type (α, ρ, β), e^* is of adjoint type $(\beta, \bar\rho, \alpha)$. The operator adjoint preserves the localization.

Therefore, the C^*-subalgebra $\mathcal{F}_{\rm red} \subset \mathcal{B}(\mathcal{H}_{\rm red})$ generated by the operators $F(e, A)$ is densely spanned by $F(e, A)$. The localized subalgebras $\mathcal{F}_{\rm red}(\mathcal{O})$ are taken as the von Neumann algebras generated by localized $F(e, A)$.

For the CPT theorem we shall need another conjugation operation which in contrast to the adjoint preserves (actually, charge conjugates) the source and range sectors. To this aim, we combine the adjoint with the linear operator "reversal" defined by

$$F(e,\,A)\hat{}:=\sqrt{d_\beta/d_\alpha}\cdot\sum_{\hat e}\theta^e_{\hat e}\,F(\hat e,\,A)\,. \tag{3.10}$$

Here, if e is of type $(\alpha,\,\rho,\,\beta)$, $\hat e$ is of the "reversed" type $(\bar\beta,\,\rho,\,\bar\alpha)$. The specific choice (3.10) (with some arbitrariness in the definition of $\theta^e_{\hat e}$ (cf. (A.3)) making its appearance below in the square roots $\sqrt{\kappa_\rho}$) is motivated by the following algebraic properties.

Lemma 3.5. *The linear* operator reversal $\hat{}$ *preserves the localization and commutes with the adjoint* $*$. *It is an involution up to an intrinsic sign if pseudoreal sectors are involved (cf. (3.2)):*

$$(\widehat{\hat F})=\frac{\chi_\beta}{\chi_\alpha}\cdot F\,. \tag{3.11}$$

Corollary 3.6. *The antilinear* charge conjugation *operation*[3]

$$\overline{F}:=(\hat F)^*=(\widehat{F^*})=\sqrt{d_\alpha/d_\beta}\,(d_\rho/\chi_\rho)\cdot\sum_{\hat e}\zeta_{e\bar e}\,F(\bar e,\,\bar\rho(A^*)R_\rho) \tag{3.12}$$

preserves the localization. It involves conjugation of source, charge, and range: if e is of type $(\alpha,\,\rho,\,\beta)$, then $\hat e$ is of conjugate type $(\bar\alpha,\,\bar\rho,\,\bar\beta)$. It is an involution up to a phase:

$$\overline{(\overline{F})}=\frac{\chi_\beta}{\chi_\alpha}\cdot F\,. \tag{3.13}$$

3.4–3.6 can be verified by direct computation (with repeated use of the formulae of Sec. 2.2., and especially the identities (A.4)).

Along with these algebraic conjugations we have the following "global" (as opposed to (3.5)) commutation relations.

Proposition 3.7. (*Weak Locality*) *Let $F_n=F(e_n,\,A_n)$ be a chain of reduced field bundle operators of charge $c(e_i)=\rho_i$, such that $s(e_1)=r(e_n)=[0]$, i.e. $F_n\ldots F_1$ interpolates from the vacuum sector to the vacuum sector. Let F_n be localized in regions $\mathcal{O}_i$. Then*

$$F_n\ldots F_1=\sqrt{1/\Pi_i\kappa_i}\cdot\hat F_1\ldots\hat F_n\quad\text{if}\quad\mathcal{O}_1<\ldots<\mathcal{O}_n\,. \tag{3.14}$$

The factor is inverted if the ordering of the localizations is inverted.

This is a precise generalization to bounded exchange operators of Jost's Weak Locality property [31].

Corollary 3.8. *Let $F_n=F(e_n,\,A_n)$ be a chain of reduced field bundle operates of charge $c(e_i)=\rho_i$, such that $s(e_1)=[0]$, i.e. $F_n\ldots F_1$ interpolates from the vacuum*

[3] We adopt this term here although it does *not* match with the conventional notion of (linear) charge conjugation of Wightman fields.

sector to some sector $r(e_n) = [\rho]$. *Let* F_n *be localized in regions* $\mathcal{O}_i$. *Then*

$$\overline{F_n \dots F_1} = \sqrt{(\Pi_i \kappa_i)/\kappa_\rho} \cdot \overline{F}_n \dots \overline{F}_1 \quad \textit{if} \quad \mathcal{O}_1 < \dots < \mathcal{O}_n\,. \tag{3.15}$$

The factor is inverted if the ordering of the localizations is inverted.

We intend to consider below limits of reduced field bundle elements with decreasing localization intervals. The preceding propositions remain valid for such point-like fields, and when supplemented with the appropriate space-time covariance and spectrum condition, give rise to the CPT theorem by arguments similar to [31, 32] or [33]. Note that the reversal $\hat{}$ does not respect the algebra (3.4). Therefore, the charge conjugation (3.12) is not an anti-homomorphism of the reduced field bundle, nor is it a homomorphism as (3.15) shows. Yet, with the substitute of the anti-homomorphism property given in the following lemma it seems powerful enough to play the role of the operator adjoint in a generalized modular (Tomita-Takesaki) theory. The operator adjoint cannot be used to define a modular conjugation w.r.t. the reduced field bundle because of the source and range prescriptions.

Lemma 3.9. *Let* $F_i = F(e_i, A_i)$ *be two reduced field bundle operators of charge* ρ_i, *and* $s(e_1) = [0]$, $r(e_2) = [\rho]$. *Then*

$$\overline{F(e_2, A_2)F(e_1, A_1)} = \sum_{f_1 \circ f_2} \sqrt{\kappa_1\kappa_2/\kappa_\rho}\, R^{f_1\circ f_2}_{e_2\circ e_1}(+) \cdot \overline{F(f_1, A_1)} \quad \overline{F(f_2, A_2)}\,. \tag{3.16}$$

This formula holds irrespective of the localization of F_i. The coefficient matrix equals $\sqrt{\kappa_\rho/\kappa_1\kappa_2}\, R^{f_1\circ f_2}_{e_2\circ e_1}(-)$ by virtue of (3.7), and has square **1**. The "off-vacuum" generalization of (3.16) is more complicated; one has to replace the κ-factor by the sum over the projection matrices in path space pertaining to the different fusion channels $[\rho]$ contributing to $[\rho_1\rho_2]$, each multiplied with the corresponding κ-factor.

Proof of 3.7. For $\mathcal{O}_1 < \dots < \mathcal{O}_n$ we have by repeated use of the commutation relations

$$\begin{aligned} &F(e_n, A_n)\dots F(e_1, A_1) \\ &\quad = \sum_{f_i} (T^*_{e_n} \dots T^*_{e_1} \varepsilon^{(n)}(\rho_n, \dots, \rho_1) T_{f_n} \dots T_{f_1}) \cdot F(f_1, A_1) \dots F(f_n, A_n)\,, \end{aligned}$$

where $\varepsilon^{(n)}(\dots) := \varepsilon(\rho_2 \dots \rho_n, \rho_1)\varepsilon(\rho_3 \dots \rho_n, \rho_2) \dots \varepsilon(\rho_{n-1}\rho_n, \rho_{n-2})\varepsilon(\rho_n, \rho_{n-1})$ is the statistics operator corresponding to the re-ordering of the field operators. We shall compute the matrix elements. Because $s(f_n) = [0] = r(e_n)$ and $c(f_i) = c(e_i)$, f_n is of trivial type and equals $\hat{e}_n$. Therefore, $T_{f_n} = 1$ and $\theta^{e_n}_{\hat{e}_n} = 1$. Call $T = \varepsilon(\rho_n, \rho_{n-1})T_{f_{n-1}}$, and let $[\rho] = r(f_{n-1})$. Repeated use of (2.2) gives

$$\varepsilon^{(n)}(\rho_n, \dots, \rho_n)T_{f_{n-1}} = \rho_1 \dots \rho_{n-2}(T)\varepsilon^{(n-1)}(\rho, \rho_{n-2}, \dots, \rho_1)\,.$$

Next, let $[\sigma] = r(e_{n-2})$. Then

$$T^*_{e_n} \dots T^*_{e_1}\rho_1 \dots \rho_{n-2}(T) = T^*_{e_n} T^*_{e_{n-1}} \sigma(T) T^*_{e_{n-2}} \dots T^*_{e_1}\,.$$

Inserting a complete basis $\Sigma_g T_g T_g^*$ after $\sigma(T)$, and observing that $T_{e_n}^* T_{e_{n-1}}^* \sigma(T) T_g$ is an intertwiner from $r(g)$ to $r(e_n) = [0]$, we conclude that only $r(g) = [0]$ can contribute, and hence $\sigma = \bar{\rho}$ and f_{n-1} is of type $\hat{e}_{n-1}$. Therefore, $T_{e_n} = R_{\rho_n}$ and $T_g = R_\rho$, and the mentioned self-intertwiner of the vacuum equals $\theta_{\hat{e}_{n-1}}^{e_{n-1}}$ except for the κ-phases in the first of its definitions (A.3). The remaining expression is of the same form as the original matrix element with n decreased by 1. Repeating the argument, we arrive at the conclusion (3.14). For the reversed ordering of localizations, use the equivalent second definition of θ in (A.3), yielding the inverse κ-factor.

Proof of 3.8. By (3.4), $F_n \dots F_1$ is some element $F = F(e, A)$ with e of trivial type $(0, \rho, \rho)$. Pick an element $G = F(e^*, B)$ localized in $\mathcal{O} > \mathcal{O}_n$. Then by (3.14)

$$\kappa_\rho^{-1} \hat{F}\hat{G} = GF = GF_n \dots F_1 = \sqrt{1/\kappa_\rho \Pi_i \kappa_i} \cdot \hat{F}_1 \dots \hat{F}_n \hat{G}\,.$$

Taking the adjoint yields

$$\overline{G}\,\overline{F} = \sqrt{(\Pi_i \kappa_i)/\kappa_\rho} \cdot \overline{G}\,\overline{F}_n \dots \overline{F}_1\,.$$

Finally, choose $B = U$ a unitary intertwiner: $\bar{\rho} \to \sigma$ with σ localized in $\mathcal{O}$. Then $G = F(e^*, U)$ is localized in $\mathcal{O}$, and $\hat{G}$ inverts $\overline{G}$ on $\mathcal{H}_\rho$ up to some non-vanishing factors:

$$\hat{G}\overline{G} \propto F(\bar{e}, U) F(\hat{e}, \rho(U^*) R_{\bar{\rho}}) = \sum_{f,g} D_{f,g}^{\bar{e}\circ\hat{e}} \cdot F(g, T_f^* R_{\bar{\rho}}) \propto 1|_{\mathcal{H}_\rho}$$

since $T_f^* R_{\bar{\rho}} = 0$ unless $r(f) = [0]$, therefore $c(g) = [0]$ and $T_f = R_{\bar{\rho}}$, $T_f^* R_{\bar{\rho}} = 1$.

Proof of 3.9. By direct computation with (3.4) and (3.12), exploiting the triviality of $s(e_1)$, and hence $T_{e_1} = 1$, to evaluate the D and ζ coefficients, we get for the l.h.s.

$$\text{l.h.s.} = \sqrt{d_\rho}/\chi_\rho \cdot F(\bar{e}, \bar{\rho}(A_1^* \rho_1(A_2^*)) \cdot \bar{\rho}(T_{e_2}) R_\rho)\,,$$

with e of type $(0, \rho, \rho)$. Similarly, with $T_{f_2} = 1$ the r.h.s. yields

$$\begin{aligned}\text{r.h.s.} = \sum_{f_1 \circ f_2} &\sqrt{\kappa_\rho/\kappa_1\kappa_2}\, R_{e_2 \circ e_1}^{f_1 \circ f_2}(-) \cdot (d_1 d_2/\chi_1\chi_2\sqrt{d_\rho}) \cdot \zeta_{f_1 \bar{f}_1} \\ &\cdot F(\bar{e}, \bar{\rho}(A_1^* \rho_1(A_2^*)) \cdot T_{\bar{f}_1}^* \bar{\rho}_2(R_1) R_2)\,.\end{aligned}$$

Now, consider the intertwiner $X = T_{\bar{f}_1}^* \bar{\rho}_2(R_1) R_2 : \text{id} \to \bar{\rho}\rho_2\rho_1$ contributing to the r.h.s., which can be expanded into an orthonormal basis of intertwiners $X_f = \bar{\rho}(\varepsilon(\rho_2, \rho_1) T_f) R_\rho$ with f of the same type (ρ_1, ρ_2, ρ) as f_1:

$$X = \sum_f (X_f^* X) \cdot X_f\,.$$

The scalar product $X_f^* X$ equals $\zeta^*_{\bar{f}_1 f}$ as in (A.3) except for the factors κ. With (A.4(e, f)), the two matrices ζ yield δ_{ff_1} times some more factors. The R-matrix element of the r.h.s. multiplied with $\varepsilon(\rho_2, \rho_1)T_{f_1}$ yields just T_{e_2}, so the intertwiner in the local entry of the reduced field bundle operator on the r.h.s. becomes the same as on the l.h.s. Finally, collecting all the factors χ, κ, d one establishes equality.

4. Covariance

Let us now consider *covariant* superselection sectors. A DHR endomorphism ρ is covariant if there is an associated strongly continuous unitary positive-energy representation $\mathcal{U}_\rho : \tilde{G} \to \mathcal{B}(\mathcal{H}_0)$ of the universal covering of the space-time symmetry group G satisfying

$$\mathcal{U}_\rho(\tilde{g})\pi_0\rho(A)\mathcal{U}_\rho(\tilde{g})^* = \pi_0\rho(\alpha_g(A)), \tag{4.1}$$

where $\alpha : G \to Aut(\mathcal{A})$ is the representation of G by automorphisms of the local net $\mathcal{A}$, and $\tilde{g} \mapsto g$ is the covering projection $\tilde{G} \to G$. We assume that there is a unique vacuum vector $\Omega \equiv (0, \Omega) \in \mathcal{H}_0$ invariant under the covariance of the vacuum sector $\mathcal{U}_0(\gamma)$.

The property of being covariant does not change under unitary equivalence transformations among DHR endomorphisms, and is therefore a property of the sector. We recall from [4] that the product and the conjugate of covariant sectors are again covariant, as well as every subsector of a covariant sector. The latter statement follows by an argument given in [4], if the space-time covariance group possesses no non-trivial finite-dimensional unitary representations, e.g., the Poincaré groups in two or more dimensions, or the Möbius group. More precisely, for covariant DHR endomorphisms ρ, σ

$$\begin{aligned}\mathcal{U}_{\sigma\rho}(\tilde{g}) &= \mathcal{U}_\sigma(\tilde{g})\pi_0\sigma(X_\rho(\tilde{g})) \quad \text{where} \quad X_\rho(\tilde{g}) = \pi_0^{-1}(\mathcal{U}_0(g)^*\mathcal{U}_\rho(\tilde{g})) \\ &= \pi_0\sigma(Y_\rho(\tilde{g}))\mathcal{U}_\sigma(\tilde{g}) \quad \text{where} \quad Y_\rho(\tilde{g}) = \pi_0^{-1}(\mathcal{U}_\rho(\tilde{g})\mathcal{U}_0(g)^*),\end{aligned} \tag{4.2}$$

$$\mathcal{U}_{\bar{\rho}}(\tilde{g}) = (d_\rho/\chi_\rho)\cdot\pi_0[\bar{\rho}(\alpha_g(R_{\bar{\rho}}^*)Y_\rho(\tilde{g})^*)R_\rho]\cdot\mathcal{U}_0(g) \quad (\rho \text{ irreducible}), \tag{4.3}$$

$$\pi_0(T)\mathcal{U}_\rho(\cdot) = \mathcal{U}_\sigma(\cdot)\pi_0(T) \quad \forall T : \rho \to \sigma. \tag{4.4}$$

Here $\mathcal{U}_0 : G \to \mathcal{B}(\mathcal{H}_0)$ is the covariance of the vacuum sector, induced in the GNS representation π_0 from the G-invariant ground state. It is important to note that the cocycle $\mathcal{U}_\rho(\cdot)\mathcal{U}_0(\cdot)^*$ intertwines the representation $\pi_0 \circ \rho$ and its translate $\pi_0 \circ (\alpha^{-1} \circ \rho \circ \alpha)$. By Haag duality, it is the image under π_0 of a local intertwiner ("charge transporter"). Therefore, $X_\rho : \rho \to \alpha^{-1}\rho\alpha$ and $Y_\rho : \alpha\rho\alpha^{-1} \to \rho$ above are well-defined local operators as long as π_0 is faithful. Clearly, $X_\rho(\tilde{g})^* = Y_\rho(\tilde{g}^{-1})$.

Exercise. Check that (4.3) is a representation of $\tilde{G}$ satisfying (4.1) for $\bar{\rho}$. To prove unitarity, note that $R' := \bar{\rho}(Y_\rho^*)R_\rho$ is an isometric intertwiner: id $\to \bar{\rho}\rho'$ where $\rho' = \alpha\rho\alpha^{-1} \in [\rho]$. Choose $U : \bar{\rho} \to \alpha\bar{\rho}\alpha^{-1}$ unitary. Then $\bar{R}' := \alpha[\rho\alpha^{-1}(U^*)R_{\bar{\rho}}]$ is an isometric intertwiner : id $\to \rho'\bar{\rho}$. Now, $\bar{\rho}(\alpha(R_{\bar{\rho}}^*)Y_\rho^*)R\cdot U = \bar{\rho}(\bar{R}'^*)R'$ is a scalar of modulus $1/d_\rho$, therefore (4.3) is unitary.

The covariance is implemented on $\mathcal{H}_{\rm red}$ by the representation $\mathcal{U} = \oplus_\alpha \mathcal{U}_\alpha$.

Lemma 4.1. *The space-time covariance acts on* $\mathcal{F}_{\rm red}$ *like*

$$\alpha_{\tilde{g}}(F(e, A)) := \mathcal{U}(\tilde{g})F(e, A)\mathcal{U}(\tilde{g})^* = F(e, Y_\rho(\tilde{g})\alpha_g(A)). \tag{4.5}$$

In particular, the effect on the local degree of freedom A depends only on the charge, but not on the source and range sectors. $\alpha_{\tilde{g}}$ *commute with the adjoint, reversal, and charge conjugation operations of Sec. 3. The action is geometric:*

$$\alpha_{\tilde{g}}(\mathcal{F}_{\rm red}(\mathcal{O})) = \mathcal{F}_{\rm red}(g(\mathcal{O})). \tag{4.6}$$

(4.5) can be verified by application to an arbitrary vector in the source Hilbert space. $\alpha_{\tilde{g}}$ commute with the reversal (3.10) since they affect only the local degree of freedom of $F(e, A)$, and with the adjoint by definition (in order to verify $\alpha(F^*) = (\alpha(F))^*$ directly, (4.3) is needed). That the action is geometric, is evident from the criterion (1.3).

5. Non-Trivial Space-Time Topologies

The analysis of the preceding sections was made on the premises that: (i) the positive-energy representations of interest describe charges which are localizable in bounded regions, i.e. the restrictions to the subalgebras of causal complements of bounded regions are all unitarily equivalent to the vacuum representation; (ii) the bounded regions of space-time (double cones) form a directed set w.r.t. inclusion that have non-trivial causal complements; and (iii) Haag duality for the local algebras associated with double cones [4] holds in the vacuum representation.

We shall discuss in this section the necessary modifications, as well as consequences when the above properties have to be relaxed. The first situation of interest is that of conformal light-cone quantum field theory in two dimensions. It has been shown in [34] that in general for these models, which are originally defined on the light-cone $\mathbb{R}$, Haag duality for intervals does not hold. It fails, e.g., for the algebra generated by the energy-momentum tensor with central charge $c > 1$. Exploiting the covariance under the real Möbius group $SL(2, \mathbb{R})/\mathbb{Z}_2$, one can, however, extend the theory to a theory on the compactified light-cone S^1. Since, e.g., by the Bisognano-Wichmann theorem [33] for local algebras generated by Wightman fields, Haag duality follows for half-lines in $\mathbb{R}$, it holds by conformal covariance for proper intervals and their complements on the circle. But the set of proper intervals is not a directed set, so the global algebra has to be defined in a different way [11, 18]. We shall present this construction below.

The second case of interest is the 2 + 1-dimensional situation with charges localized along space-like cones extending to infinity. This is the best localization that may be expected *a priori* [9] for positive-energy representations with an isolated mass-shell. It suggests the physical interpretation of gauge charges, whose total space-like flux cannot vanish by Gauß' law, while the partial flux asymptotically tends to zero along any interval $< 2\pi$ of space-like directions. The case of more than 2 + 1 dimensions leading to permutation group statistics has been treated earlier [9, 24]. An extension to 2 + 1 dimensions was also performed by Fröhlich *et al.* [7], by one of us [21], and in the diploma theses of Rüger [35] and Gaberdiel [36]. Asymptotic clustering properties of the scattering matrix were discussed by one of us [37].

In the previous treatments of superselection charges localized on space-like cones S, it was observed that the operator maps ρ induced by the identification of the representation space of π with that of π_0 are not endomorphisms of $\mathcal{A}$ but only monomorphisms of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_0)$. Moreover, the unitary intertwiners relating different choices of ρ are by Haag duality in the weak closure of $\pi_0(\mathcal{A}(S))$ but, in general, not in $\pi_0(\mathcal{A})$. The necessary extension of ρ to a larger algebra depended on some arbitrary "forbidden" space-like direction r (analogous to "infinity" on the conformal light-cone), and it was the main problem to check that the resulting structure of sectors is independent of r.

Actually, there are only two points in the analysis in [9] which have to be modified in 2 + 1 dimensions. One is the Haag-Ruelle construction of multi-particle scattering states. The argument given in [9] that the multi-particle state vectors are, independently of the Lorentz frame in which the construction was performed, uniquely characterized in terms of single particle state vectors breaks down in 2 + 1 dimensions. One rather obtains, given a configuration of single particle state vectors, different multi-particle state vectors which are transformed into each other by a pure braid transformation depending on the momenta of the single particles [35, 36]. Details of the scattering theory will appear elsewhere (cf. also [7b] for the case of abelian braid group statistics).

The other point in [9] which has to be modified is the definition of field bundle elements which are localized in space-like cones. It turns out that the field bundle elements rather depend on a homotopy class of paths in the set of space-like cones and are therefore localized in a covering space. It is precisely this phenomenon which underlies braid group statistics in three dimensions.

We introduce here a more flexible formalism than in [9] to treat the algebraic situation associated with the hyperboloid of space-like directions. Actually, the situation is much the same as with the circle (the compactified conformal light-cone), and we shall present this formalism in all detail only for the latter. The essential objects are the "universal algebra" of observables $\mathcal{A}_{\text{univ}}$ containing globally localized observables, and DHR endomorphisms of $\mathcal{A}_{\text{univ}}$ which do not depend on a reference point resp. direction. The dictionary between the relevant geometry of the hyperboloid and of the circle, with complete parallelism in the algebraic structures of the universal algebra, will be given thereafter.

The following subsection is devoted to the construction of the universal algebra and of its DHR endomorphisms which induce the positive energy representations of interest. Once these structures are established, one can apply the methods of the standard DHR theory. Yet, we shall exhibit in Sec. 5.2 new algebraic structures related to global operators without a precedent in the standard case. We identify the abstract version of Verlinde's modular algebra [19] within the universal algebra. In

Sec. 5.3, we concentrate on chiral conformal theories and derive the Spin-Statistics theorem in its strong version and the CPT theorem for charged fields.

5.1. *The universal algebra and its endomorphisms*

The space-time symmetry group of two-dimensional conformal quantum field theory is $(SL(2,\mathbb{R})/\mathbb{Z}_2)\times(SL(2,\mathbb{R})/\mathbb{Z}_2)$, the product of two Möbius groups acting on either light-cone. These groups can be implemented only if the light-cone is compactified to S^1. In fact, chiral local fields such as the energy-momentum tensor are periodic in the circular coordinate ϑ, $x=\tan(\vartheta/2)$ [38].

One is therefore led to consider the family of von Neumann algebras $\mathcal{A} = (\mathcal{A}(I))_{I\in\mathcal{J}}$ on some Hilbert space $\mathcal{H}_0$, indexed by the set $\mathcal{J}$ of proper intervals $I\subset S^1$, as the theory of chiral observables. On $\mathcal{H}_0$ there is given a strongly continuous unitary positive-energy representation $\mathcal{U}$ of the Möbius group M with a unique ground state such that

$$\begin{aligned}
&\text{(i) } \mathcal{A}(I)\subset\mathcal{A}(J) \quad \text{if} \quad I\subset J\ (I,\,J\in\mathcal{J}) \quad \text{(isotony)}\\
&\text{(ii) } \mathcal{A}(I)\subset\mathcal{A}(J)' \quad \text{if} \quad I\subset J'\ (I,\,J\in\mathcal{J}) \quad \text{(locality)}\\
&\text{(iii) } \mathcal{U}(g)\mathcal{A}(I)\mathcal{U}(g^{-1})=\mathcal{A}(gI) \quad (g\in M) \quad \text{(covariance)}.
\end{aligned} \tag{5.1.1}$$

Locality will be replaced by the apparently stronger property of Haag duality

$$\mathcal{A}(I')=\mathcal{A}(I)' \quad (I\in\mathcal{J}), \tag{5.1.2}$$

where I' is the complement of I in S^1. (Note that as a consequence of the strong continuity of $\mathcal{U}$ and the covariance of $\mathcal{A}$, the algebras $\mathcal{A}(I)$ do not depend on whether or not I contains its boundary points.) Haag duality has been proven by Jörß [39] under the assumption that the multiplicities of irreducible subrepresentations of $\mathcal{U}$ are finite.

We are interested in conformally covariant positive-energy representations π of $\mathcal{A}$. By this we mean a family of representations π^I of $\mathcal{A}(I)$, $I\in\mathcal{J}$, on some Hilbert space $\mathcal{H}$ together with a strongly continuous positive-energy representation $\mathcal{U}_\pi$ of the covering group $\tilde{M}$ of M such that

$$\begin{aligned}
&\text{(i) } \quad \pi^J|_{\mathcal{A}(I)}=\pi^I \quad \text{if} \quad I\subset J \quad (I,\,J\in\mathcal{J})\\
&\text{(ii) } \mathrm{Ad}_{\mathcal{U}_\pi(\tilde{g})}\circ\pi^I=\pi^{gI}\circ\mathrm{Ad}_{\mathcal{U}(g)}|_{\mathcal{A}(I)} \quad (I\in\mathcal{J},\,\tilde{g}\in\tilde{M})
\end{aligned} \tag{5.1.3}$$

where $\tilde{g}\mapsto g$ is the covering homomorphism.

Buchholz *et al.* [5] have shown that each of these local representations π^I is unitarily equivalent to the defining representation $\mathrm{id}_{\mathcal{A}(I)} =: \pi_0^I$. As usual in the DHR theory, one exploits this equivalence for the complement of *some* $I_0\in\mathcal{J}$ to identify $\mathcal{H}_\pi$ with $\mathcal{H}_0$ such that $\pi^{I_0'}=\mathrm{id}_{\mathcal{A}(I_0')}$. We say that π is localized in I_0. If we replace I_0 by I_1 with $I_0\subset I_1$ or $I_0\supset I_1$, we obtain an equivalent representation π_1 localized in I_1

$$\pi_1^I=\mathrm{Ad}_{U_1}\circ\pi^I \quad (I\in\mathcal{J})$$

where the unitary intertwiner U_1 is an element of $\mathcal{A}(I_0' \cap I_1')'$, hence of $\mathcal{A}(I_0 \cup I_1)$ by Haag duality. Iterating this procedure, we find that the unitary intertwiner U in

$$\hat{\pi}^I = \mathrm{Ad}_U \circ \pi^I \quad (I \in \mathcal{J}) \tag{5.1.4}$$

where $\hat{\pi}$ is localized in $\hat{I}$, admits the following representation as a product of local operators. A finite sequence

$$\gamma = (I_0, I_1, \ldots, I_n = \hat{I})$$

in $\mathcal{J}$ such that for $k = 1 \ldots n$ either $I_{k-1} \subset I_k$ or $I_{k-1} \supset I_k$ may be called a path in $\mathcal{J}$ from I_0 to $\hat{I}$. Then there are local unitaries $U_k \in \mathcal{A}(I_{k-1} \cup I_k)$ $(k = 1 \ldots n)$ such that

$$U = U_n \ldots U_1 \,. \tag{5.1.5}$$

We may invert (5.1.4) and *define* π in terms of intertwiners. Namely

$$\pi^I = \mathrm{Ad}_{U^*}|_{\mathcal{A}(I)} \,, \tag{5.1.6}$$

where U of the form (5.1.5) is a unitary intertwiner to some representation $\hat{\pi}$ localized in $\hat{I}$ properly contained in I', i.e. $g\hat{I} \subset I'$ for all g in some neighbourhood of the identity in the Möbius group (Notation: $\hat{I} \Subset I'$).

In the standard theory, (5.1.6) is used for the definition of the composition of representations: for $A \in \mathcal{A}(I)$, $\pi^I(A)$ is a product of local operators, hence if π' is a representation of the algebra generated by all local operators, then the composite representation is defined by the composition of mappings

$$(\pi' \times \pi)^I := \pi' \circ \pi^I \,.$$

It is, however, not true in general, that a consistent family (in the sense of (5.1.3(i))) of local representations π'^I extends to a representation π' of this algebra. This may heuristically be understood by the possible existence of local charge operators associated to complementary regions which add in the vacuum sector to zero. Hence if π' were well-defined, the charge would assume the value zero also in the sector of π'.

There are several ways out which were adapted to models in conformal quantum field theory [5, 27, 40] or were used in the corresponding problem with gauge charges [9, 24, 7, 35, 36]. Here we take a point of view first proposed in [11] and further developed in [18]. The system of local algebras $(\mathcal{A}(I))_{I\in\mathcal{J}}$ uniquely determines a C^*-algebra $\mathcal{A}_{\mathrm{univ}}$ which satisfies the following universality condition:

(i) there are unital embeddings $i^I : \mathcal{A}(I) \to \mathcal{A}_{\mathrm{univ}}$ such that

$$i^J|_{\mathcal{A}(I)} = i^I \quad \text{if} \quad I \subset J \quad (I, J \in \mathcal{J}) \,, \tag{5.1.7}$$

and $\mathcal{A}_{\mathrm{univ}}$ is generated by the algebras $i^I(\mathcal{A}(I))$, $I \in \mathcal{J}$;

(ii) for every consistent family (in the sense of (5.1.3(i))) of representations $\pi^I : \mathcal{A}(I) \to \mathcal{B}(\mathcal{H}_\pi)$ there is a unique representation π of $\mathcal{A}_{\text{univ}}$ in $\mathcal{H}_\pi$ such that

$$\pi \circ i^I = \pi^I . \tag{5.1.8}$$

The Möbius group acts by automorphisms α_g of $\mathcal{A}_{\text{univ}}$

$$\alpha_g \circ i^I = i^{gI} \circ \text{Ad}_{\mathcal{U}(g)}|_{\mathcal{A}(I)} , \tag{5.1.9}$$

and a conformally covariant positive-energy representation $\pi = (\pi^I)_{I\in\mathcal{J}}$ in the sense of (5.1.3) corresponds to a covariant positive-energy representation π of $\mathcal{A}_{\text{univ}}$ in the usual sense. The defining representations $i^I(A) \mapsto A$ induce the vacuum representation π_0 of A_{univ}. We shall see that π_0 is in general not faithful.

Let now π be a conformally covariant positive-energy representation of $\mathcal{A}_{\text{univ}}$ which is localized in $I_0 \in \mathcal{J}$. We want to find an endomorphism ρ of $\mathcal{A}_{\text{univ}}$ such that $\pi = \pi_0 \circ \rho$. In view of the universality property of $\mathcal{A}_{\text{univ}}$ it suffices to define a family of homomorphisms $\rho^I : i^I(\mathcal{A}(I)) \to \mathcal{A}_{\text{univ}}$ such that

$$\rho^J|_{\mathcal{A}(I)} = \rho^I \quad \text{if} \quad I \subset J \quad (I,\, J \in \mathcal{J}) . \tag{5.1.10}$$

Let $I \in \mathcal{J}$ and let U be a unitary intertwiner from π to $\hat{\pi}$ where $\hat{\pi}$ is localized in $\hat{I} \in \mathcal{J}$, $\hat{I} \Subset I'$. Let $\gamma = (I_0, I_1, \dots, I_n = \hat{I})$ be a path in $\mathcal{J}$ from I_0 to $\hat{I}$, and let (5.1.5) be the corresponding factorization of U. Then we set

$$\rho^I := \text{Ad}_V|_{\mathcal{A}(I)} \quad \text{with} \quad V = i^{I_0\cup I_1}(U_1^*) \dots i^{I_{n-1}\cup I_n}(U_n^*) . \tag{5.1.11}$$

By standard arguments exploiting Haag duality it is clear that ρ^I thus defined is a consistent family (5.1.10) and do not depend on the choice of U nor on the choice of factors $U_k \in \mathcal{A}(I_{k-1} \cup I_k)$. It is less clear that it is also independent of the choice of the path γ.

A path $\tilde{\gamma}$ may be called a small deformation of γ if it is obtained by insertion of one interval between I_{k-1} and I_k for some $k = 1 \dots n$ or by omission of one interval I_k, $k = 1 \dots n-1$. It is easy to see that small deformations do not change ρ^I.

Next, $\tilde{\gamma}$ may be called homotopic to γ if it is obtained from γ by a finite sequence of small deformations. Then ρ^I can depend at most on the homotopy class of γ.

In the last step, we convince ourselves that even this can be ruled out. For this purpose it is sufficient to consider the closed paths of the form

$$\hat{\gamma} = (\hat{I},\, J_1,\, I,\, I_2,\, \hat{I}) \quad \text{where} \quad J_1,\, J_2 \Supset I \cup \hat{I}$$

from $\hat{I}$ to $\hat{I}$, and verify that ρ^I does not change if in (5.1.11) γ is replaced by $\hat{\gamma} \circ \gamma$ running through $\hat{\gamma}$ after γ. Let $\hat{U}$ be a unitary intertwiner from $\hat{\pi}$ to $\check{\pi}$ to where $\check{\pi}$ is localized in I. Then a factorization (5.1.5) of the trivial intertwiner 1 associated with $\hat{\gamma}$ is

$$1 = 1 \cdot \hat{U}^* \cdot 1 \cdot \hat{U} = \hat{U}^* \cdot \hat{U} .$$

Let $A \in \mathcal{A}(I)$. Choose some $\check{I} \Supset I$, $\check{I} \subset J_1 \cap J_2$. Then we can compute for both $J = J_1$ or J_2

$$\mathrm{Ad}_{i^J(\hat{U})}(i^I(A)) = i^J(\mathrm{Ad}_{\hat{U}}(A)) = i^J \circ \check{\pi}^I(A) = i^{\check{I}} \circ \check{\pi}^I(A),$$

where we have used (i) that $\hat{U}$ and A lie in both $\mathcal{A}(J_1)$ and $\mathcal{A}(J_2)$, (ii) that $\mathrm{Ad}_{\hat{U}}(A) = \check{\pi}^I(A)$, and (iii) that $\check{\pi}^I(A)$ is contained in $\mathcal{A}(\check{I})$. Since the result is the same in both cases, we have

$$\mathrm{Ad}_V(i^I(A)) = i^I(A) \quad \text{where} \quad V = i^{J_1}(\hat{U})^* i^{J_2}(\hat{U}).$$

We conclude that $\rho^I : i^I(\mathcal{A}(I)) \to \mathcal{A}_{\mathrm{univ}}$ given by (5.1.11) are completely independent of the path γ and of the local unitaries U_k. Since $\mathcal{A}_{\mathrm{univ}}$ is generated by $(i^I(\mathcal{A}(I)))_{I \in \mathcal{J}}$, these homomorphisms define a localized DHR endomorphism $\rho : \mathcal{A}_{\mathrm{univ}} \to \mathcal{A}_{\mathrm{univ}}$ such that $\pi = \pi_0 \circ \rho$.

Let us now turn to the analogous situation in 2 + 1-dimensional theories with charges localized in space-like cones. The latter are subsets of Minkowski space $\mathbb{M}^3$ of the form

$$S = a + \bigcup_{\lambda > 0} \lambda \mathcal{O},$$

where $a \in \mathbb{M}^3$ is the apex and $\mathcal{O}$ is a double-cone of space-like directions:

$$\mathcal{O} = \{ r \in \mathbb{M}^3 | r^2 = -1 \quad \text{and} \quad r_+ - r,\ r - r_- \in V_+ \}$$

with $r_+^2 = r_-^2 = -1$ and $r_+ - r_- \in V_+$. By $\mathcal{S}$ we denote the set of space-like cones. The causal complement in $\mathbb{M}^3$ of a space-like cone S is denoted by S'.

We consider theories where the vacuum representation π_0 satisfies Haag duality for space-like cones:

$$\pi_0(\mathcal{A}(S'))' = \pi_0(\mathcal{A}(S))'' \quad (S \in \mathcal{S}), \tag{5.1.12}$$

and representations π of $\mathcal{A}$ which in restriction to causal complements of all space-like cones are equivalent to π_0:

$$\pi|_{\mathcal{A}(S')} \simeq \pi_0|_{\mathcal{A}(S')} \quad (S \in \mathcal{S}). \tag{5.1.13}$$

The selection criterion of finite statistics may be replaced in view of [6] by the condition of "almost Haag duality", i.e. the Jones indices of the inclusions

$$\pi(\mathcal{A}(S))'' \subset \pi(\mathcal{A}(S'))' \quad (S \in \mathcal{S})$$

are finite. The criterion (5.1.13) has been proven to be fulfilled (for a suitable vacuum representation π_0) under the assumption that π contains states of a single particle with an isolated mass-shell [9], and then π satisfies almost Haag duality provided π_0 satisfies Haag duality (5.1.12).

We may literally repeat the above definitions and arguments for the 2 + 1-dimensional situation, if we let the index $I \in \mathcal{J}$ of subalgebras run over the set $\mathcal{J} = \mathcal{S} \cup \{S' | S \in \mathcal{S}\}$ containing the space-like cones and their causal complements in $\mathbb{M}^3$. The associated algebras $\mathcal{A}(S)$, $\mathcal{A}(S')$ are, in accordance with (5.1.13), identified with their weak closures in the vacuum representation. The previous local intertwiners are understood as elements of some $\mathcal{A}(S)$ or $\mathcal{A}(S')$. Clearly, the Möbius group has to be replaced by the Poincaré group.

We have therefore also established the universal algebra $\mathcal{A}_{\text{univ}}$ and its DHR endomorphisms for three-dimensional quantum field theory.

We emphasize that by the above construction, and in contrast to previous treatments in 2 + 1 dimensions, the universal algebra and its endomorphisms are defined globally, i.e. without reference to an auxiliary direction. Thus the observables and their charges "live" in "true" space-time. Only the localization of charged *fields* will be seen below to depend on an auxiliary direction resp. "point at infinity" of S^1, and therefore the latter live on a covering space.

We expect that the present geometrical formalism is sufficiently flexible to apply also to more general space-times than the two treated here.

5.2. *The center of the universal algebra and its representations*

We shall in this subsection discuss the implications of the existence of global intertwiners in the universal algebra $\mathcal{A}_{\text{univ}}$, and in particular establish the relation between the statistics monodromy and the non-faithfulness of the vacuum representation. Again, the results of this subsection hold as well in 2 + 1 dimensions, with the appropriate replacements as described in the preceding subsection.

Once the existence of the universal algebra and its endomorphisms has been established, it is more convenient in the following to consider $A \in \mathcal{A}_{\text{univ}}$ and $\mathcal{A}(I) \subset \mathcal{A}_{\text{univ}}$ as *abstract* elements of $\mathcal{A}_{\text{univ}}$, and only $\pi(A)$ as operators on the Hilbert space $\mathcal{H}_0$. None of these representations is globally faithful. Then the unital embeddings i^I (5.1.7) become obsolete, and are replaced by the pre-images $(\pi_0|_{\mathcal{A}(I)})^{-1}$ of the *local* vacuum representations.

It is also convenient to introduce for every "point at infinity" $\xi \in S^1$, the C^*-subalgebra $\mathcal{A}_\xi = \overline{\{\mathcal{A}(I) | I \subset S^1,\ \xi \notin I\}}$. Endomorphisms of $\mathcal{A}_{\text{univ}}$ localized in $I \not\ni \xi$ are DHR endomorphisms of $\mathcal{A}_\xi$. Indeed, the theory of statistics and the reduced field bundle can be established on $\mathcal{A}_\xi$ without modification, since for the existence of *local* intertwiners between endomorphisms of $\mathcal{A}_\xi$ Haag duality on S^1 is sufficient. Restricted to $\mathcal{A}_\xi$, the vacuum representation π_0 is faithful.

However, the statistics operator defined in Sec. 2.2. will in general depend on the reference point at infinity ξ, since the "initial condition" (2.5) does. More precisely, for ρ localized in J and σ in I, $\varepsilon_\xi(\rho, \sigma) \in \mathcal{A}_\xi$ is defined as in Sec. 2 provided $\xi \in I' \cap J'$, and does not change as long as ξ changes *continuously*. Therefore, when $I' \cap J'$ is disconnected, the statistics operator may take different values. The following configuration is illustrative: Let I, J be disjoint intervals and ξ, ζ points in the two connected components of $I' \cap J'$ such that ζ is reached from I by a positive rotation, and ξ by a negative one. Let ρ be localized in J and σ in I. Both of them are endomorphisms of both $\mathcal{A}_\xi$ and $\mathcal{A}_\zeta$. But by (2.5), within $\mathcal{A}_\xi$, $\varepsilon_\xi(\rho, \sigma)$ is trivial, while within $\mathcal{A}_\zeta$, $\varepsilon_\zeta(\sigma,\rho)$ is trivial. The prescriptions ε_ξ and ε_ζ can coincide only if the monodromy is trivial.

Next, let us study global intertwiners. Let $I,\ J \in \mathcal{J}$ and $\xi,\ \zeta \in I' \cap J'$ as above, and let this time both ρ and σ be localized in I, and choose $\hat\rho \in [\rho]$ localized in J. For this geometry, the statistics operators $\varepsilon(\rho, \sigma)$ and $\varepsilon(\sigma, \rho) \in \mathcal{A}(I) \subset \mathcal{A}_\xi \cap \mathcal{A}_\zeta$ coincide for ξ and ζ. By Haag duality, a unitary intertwiner $\mathcal{V} : \pi_0\rho \to \pi_0\hat\rho$ lies both in $\pi_0(\mathcal{A}_\xi)$ and in $\pi_0(\mathcal{A}_\zeta)$. Let $V_+ \in \mathcal{A}_\xi$ and $V_- \in \mathcal{A}_\zeta$ be its pre-images: $\pi_0(V_+) = \pi_0(V_-) = \mathcal{V}$. We want to study the global self-intertwiners

$$V_\rho = V_+^* V_- : \rho \to \rho\,. \tag{5.2.1}$$

V_ρ are in fact independent of $\mathcal{V}$ and $\hat\rho \in [\rho]$, and commute with local self-intertwiners of ρ. (*Proof*: Let U be a unitary local self-intertwiner of ρ. Then $\mathcal{V}\pi_0(U)$ is a different choice instead of $\mathcal{V}$, giving rise to $V_\pm U = V_\pm U V_\pm^* \cdot V_\pm$ instead of $V_\pm$. But $V_\pm U V_\pm^*$ are *local* self-intertwiners of $\hat\rho$ with the same images in π_0, and therefore coincide and cancel from (5.2.1). Replacing $\hat\rho$ by $\mathrm{Ad}_{\hat U} \circ \hat\rho$ and $\mathcal{V}$ by $\pi_0(\hat U)\mathcal{V}$ trivially does not change (5.2.1).)

From the definition of statistics operators, and because J lies to the right of I w.r.t. ξ, and to the left of I w.r.t. ζ, the statistics operators can be computed from the local charge transporters $V_\pm$:

$$\varepsilon(\rho, \sigma) = \sigma(V_+)^* V_+\,, \quad \varepsilon(\sigma, \rho)^* = \sigma(V_-)^* V_-\,, \tag{5.2.2}$$

and therefore

$$\sigma(V_\rho) = \varepsilon(\rho, \sigma) V_\rho \varepsilon(\sigma, \rho) \quad \Rightarrow \quad \pi_0\sigma(V_\rho) = \pi_0[\varepsilon(\rho, \sigma)\varepsilon(\sigma, \rho)]\,. \tag{5.2.3}$$

The first of these identities stands in contrast to (2.2) which therefore in general does not apply to global intertwiners. The second identity exhibits the non-faithfulness of the vacuum representation: while V_ρ is trivially represented by π_0, its value in $\pi_0\sigma$ is the monodromy operator.

Proposition 5.1. *The (global) "Casimir operators"*

$$W_\rho := R_{\bar\rho}^* V_\rho R_{\bar\rho} \quad (\equiv \kappa_\rho^2 \cdot \phi_\rho(V_\rho) \quad \text{if } \rho \text{ is irreducible}) \tag{5.2.4}$$

(with $R_{\bar\rho} : id \to \rho\bar\rho$ as in Sec. 2.3.) lie in the center $\mathcal{A}_{\mathrm{univ}} \cap \mathcal{A}'_{\mathrm{univ}}$ of the universal algebra. They depend only on the class $[\rho]$. Their values in different representations are (for $\rho,\ \sigma$ irreducible)

$$\pi_0\sigma(W_\rho) = \pi_0\sigma(W_{\bar\rho})^* = \pi_0\rho(W_\sigma) = \sum_\tau N_{\sigma\rho}^\tau \frac{\kappa_\tau}{\kappa_\rho\kappa_\sigma} \frac{d_\tau}{d_\rho d_\sigma}\,. \tag{5.2.5}$$

The central elements $C_\rho := d_\rho \cdot W_\rho$ satisfy the fusion algebra

$$\begin{aligned} &\text{(a)}\ C_\rho^* = C_{\bar\rho} \\ &\text{(b)}\ C_{\sigma\rho} = C_\sigma \cdot C_\rho \\ &\text{(c)}\ C_\rho = \sum_\alpha N^\alpha C_\alpha \quad \text{if} \quad \rho \simeq \oplus_\alpha N^\alpha \rho_\sigma\,. \end{aligned} \tag{5.2.6}$$

Proof. For fixed choice of ρ and $\bar\rho$, $R_{\bar\rho}$ is unique up to $U\,\rho(\bar U)$, where U and $\bar U$ are unitary local self-intertwiners of ρ and $\bar\rho$. This unitary operator commutes with V_ρ and does not change W_ρ. Changing ρ and $\bar\rho$ with arbitrary local unitaries into $\mathrm{Ad}_U \circ \rho$ and $\mathrm{Ad}_{\bar U} \circ \bar\rho$, one may replace $R_{\bar\rho}$ by $U\,\rho(\bar U)R_{\bar\rho}$ and V_ρ by $\mathrm{Ad}_U V_\rho$ which again leaves W_ρ invariant. The formula for irreducible ρ follows from (2.7), (3.7), and (5.2.3), if one chooses $R_{\bar\rho} = \varepsilon(\bar\rho,\,\rho)R_\rho$. That W_ρ are central, is evident from the intertwining properties. In order to derive (5.2.5), we compute

$$\begin{aligned}\pi_0\sigma(W_\rho) &= \pi_0\sigma(R^*_{\bar\rho}V_\rho R_{\bar\rho}) = \pi_0[\sigma(R^*_{\bar\rho})\varepsilon(\rho,\,\sigma)\varepsilon(\sigma,\,\rho)\sigma(R_{\bar\rho})] \\ &= \pi_0[R^*_{\bar\rho}\rho(\varepsilon(\bar\rho,\,\sigma)^*\varepsilon(\sigma,\,\bar\rho)^*)R_{\bar\rho}] = \pi_0[\phi_{\bar\rho}(\varepsilon(\bar\rho,\,\sigma)^*\varepsilon(\sigma,\,\bar\rho)^*)]\,.\end{aligned}$$

For irreducible ρ, σ, the local self-intertwiner of σ in $[\ldots]$ is a scalar and can be rewritten

$$\sigma(R^*_{\bar\rho})\varepsilon(\rho,\,\sigma)\varepsilon(\sigma,\,\rho)\sigma(R_{\bar\rho}) = R^*_{\bar\rho}\cdot\phi_\sigma(\varepsilon(\rho,\,\sigma)\varepsilon(\sigma,\,\rho))\cdot R_{\bar\rho} = \phi_\sigma(\varepsilon(\rho,\,\sigma)\varepsilon(\sigma,\,\rho))\,.$$

Inserting orthonormal bases $1 = \Sigma_e T_e T_e^*$ in either of these final expressions, and evaluating the monodromy by (3.7) and $\phi(T_eT_e^*)$ by (3.8) yields (5.2.5).

To prove (5.2.6(b)), we may choose unitary charge transporters for $\sigma\rho$ by $V_{\sigma\rho+} = V_{\sigma+}\sigma(V_{\rho+}) = V_{\sigma+}V_{\sigma+}\varepsilon(\rho,\,\sigma)^*$ and $V_{\sigma\rho-} = V_{\sigma-}V_{\rho-}\varepsilon(\rho,\,\sigma)^*$. Then

$$V_{\sigma\rho} = \sigma(V^*_{\rho+})V^*_{\sigma+}V_{\sigma-}V_{\rho-}\varepsilon(\rho,\,\sigma)^* = V_\sigma\varepsilon(\rho,\,\sigma)V_\rho\varepsilon(\rho,\,\sigma)^*\,,$$

and (b) follows (with the choice $R_{\overline{\sigma\rho}} = \sigma(R_{\bar\rho})R_{\bar\sigma}$) from the definition and the identity $\varepsilon(\rho,\,\sigma)^*\sigma(R_{\bar\rho}) = \rho(\varepsilon(\bar\rho,\,\sigma))R_{\bar\rho}$.

For (c), let ρ_α be localized in I and let $T:\rho_\alpha \to \rho$ be a local intertwiner. Then $V_{\rho\pm}TV^*_{\alpha\pm}$ are local intertwiners : $\hat\rho_\alpha \to \hat\rho$ with the same images in π_0 for both signs, and therefore coincide. We conclude that T intertwines the global operators:

$$TV_\alpha = V_\rho T\,.$$

Next, choose orthonormal bases of local intertwiners $T_{\alpha,i}:\rho_\alpha \to \rho$ and $T_{\bar\alpha,j}:\rho_{\bar\alpha} \to \bar\rho$ $(i,\,j = 1\ldots N^\alpha)$. Then $\{\sqrt{d_\rho/d_\alpha}\cdot T^*_{\alpha,i}R_{\bar\rho}\}$ and $\{\rho_\alpha(T_{\bar\alpha,j})R_{\bar\alpha}\}$ are two sets of orthonormal bases of the local intertwiner spaces : $\mathrm{id} \to \rho_\alpha\bar\rho$. We compute

$$\begin{aligned}C_\rho = d_\rho\cdot R^*_{\bar\rho}V_\rho R_{\bar\rho} &= d_\rho\cdot R^*_{\bar\rho}\Sigma_{\alpha,i}T_{\alpha,i}V_\alpha T^*_{\alpha,i}R_{\bar\rho} \\ &= \Sigma_{\bar\alpha,j}d_\alpha\cdot R^*_{\bar\alpha}\rho_\alpha(T^*_{\bar\alpha,j})V_\alpha\rho_\alpha(T_{\bar\alpha,j})R_{\bar\alpha} = \Sigma_\alpha N^\alpha C_\alpha\,.\end{aligned}$$

Finally, in order to prove (a), let ρ and $\bar\rho$ be localized in I. Choose equivalent $\hat\rho$ and $\hat{\bar\rho}$ localized in J, and $R_{\hat{\bar\rho}}:\mathrm{id} \to \hat\rho\hat{\bar\rho}$ local intertwiners in $\mathcal{A}(J)$ as in Sec. 2.3. For $V_\pm$ unitary charge transporters : $\rho \to \hat\rho$ with the same images in π_0 as before, we choose $\bar V_+ = d_\rho\cdot R^*_\rho\bar\rho(V^*_+R_{\hat{\bar\rho}}) = d_\rho\cdot R^*_\rho\varepsilon(\rho,\,\bar\rho)V^*_+R_{\hat{\bar\rho}}$ and $\bar V_- = d_\rho\cdot R^*_\rho\varepsilon(\rho,\,\bar\rho)V^*_-R_{\hat{\bar\rho}}$ which are unitary charge transporters : $\bar\rho \to \hat{\bar\rho}$ with the same images in π_0. Then

$$\begin{aligned}W_{\bar\rho} = R^*_\rho\bar V^*_+\bar V_-R_\rho &= d^2_\rho\cdot R^*_\rho R^*_{\hat{\bar\rho}}V_+\varepsilon(\rho,\,\bar\rho)^*R_\rho R^*_\rho\varepsilon(\rho,\,\bar\rho)V^*_-R_{\hat{\bar\rho}}R_\rho \\ &= d^2_\rho\cdot R^*_{\hat{\bar\rho}}V_+\rho(R^*_\rho)R_{\bar\rho}R^*_{\bar\rho}\rho(R_\rho)V^*_-R_{\hat{\bar\rho}} = R^*_{\hat{\bar\rho}}V_+V^*_-R_{\hat{\bar\rho}} = W^*_{\hat\rho} = W^*_\rho\,.\end{aligned}$$

In this calculation we have used (i) that $V_{\pm}^{*}R_{\hat{\hat{\rho}}}$ are intertwiners: id $\rightarrow \rho\hat{\hat{\rho}}$ and $\hat{\hat{\rho}}$ acts trivially on R_ρ, (ii) that we may choose $R_\rho = \varepsilon(\rho, \bar{\rho})R_{\bar{\rho}}$, (iii) that $\rho(R_\rho^*)R_{\bar{\rho}}R_{\bar{\rho}}^*\rho(R_\rho) = d_\rho^{-2}$, and (iv) the fact that since V_+ is a charge transporter of ρ to the right, V_+^* is a charge transporter for $\hat{\rho}$ to the left and vice versa: $\hat{V}_\pm = V_\mp^*$. This completes the proof.

Let us now assume for the moment the number of irreducible superselection sectors of the theory to be finite. The matrix (5.2.5) of numerical values $\pi_0\sigma(W_\rho)$ (as $[\rho]$, $[\sigma]$ run over the sectors) was discussed previously by one of us [20] and by Fröhlich *et al.* [7c]. It was found that in theories with finitely many sectors the values of W_ρ in all representations $\pi_0\sigma$ uniquely characterize the sector $[\rho]$, provided the vacuum sector is the only one which has trivial monodromy with all other sectors (the "non-degenerate" case). In other words, the central elements W_ρ may be regarded as a complete system of "charge operators". Observe that (5.2.6) generalize the algebra satisfied by the sums over conjugacy classes of a finite gauge group represented in Hilbert space, and therefore the matrix $(\pi_0\sigma(W_\rho))$ is a generalized (self-dual) character table. We recall its properties in

Corollary 5.2. ([*20, 7c*]) *The matrix (5.2.5) is invertible*

(i) *iff the theory is non-degenerate, or*

(ii) *iff* $|\Sigma_\rho\kappa_\rho d_\rho^2|^2 = \Sigma_\rho d_\rho^2$. *The matrices*

$$\begin{aligned} S_{\rho\sigma} &:= |\Sigma_\gamma d_\gamma^2|^{-1/2} d_\rho d_\sigma \cdot \pi_0\sigma(W_{\bar{\rho}}) \\ T &= \kappa^{-1}\cdot \mathrm{Diag}(\kappa_\rho) \quad \text{where} \quad \kappa^3 = (\Sigma_\rho\kappa_\rho d_\rho^2)/|\Sigma_\rho\kappa_\rho d_\rho^2| \end{aligned} \tag{5.2.7}$$

satisfy Verlinde's modular algebra [19]

$$\begin{aligned} &(a)\ SS^\dagger = TT^\dagger = \mathbb{1} \\ &(b)\ TSTST = S \\ &(c)\ S^2 = C \\ &(d)\ TC = CT = T\,, \end{aligned} \tag{5.2.8}$$

where $C_{\rho\sigma} = \delta_{\bar{\rho}\sigma}$ *is the charge conjugation matrix.*

This algebra was originally found in models of conformal quantum field theory [19]. In [20, 7c] this remarkable result was derived from the DHR theory without any input of conformal covariance. It holds also for theories in 2 + 1 dimensions with finitely many sectors (cf. also App. B, remark after Cor. B.3).

Finally, the central element $W/\pi_0(W)$ with $W = \sum_\rho \kappa_\rho^{-1}d_\rho^2$. W_ρ can be seen to take the value κ_σ in the representation $\pi_0\sigma$. Anticipating the Spin-Statistics theorem 5.4 below, this coincides with the operator implementing rotations by 2π. Therefore, in the center of the universal algebra of *observables* are united special elements of both the space-time *covariance group* and the unknown quantum *symmetry algebra.* This is clearly a structural departure from the conventional situation with a global gauge group.

Having outlined the characteristic new features of the universal algebra and its DHR representations, we turn to the aspects of space-time covariance. Without further input, we can derive a connection between spin and statistics.

The covariance property (4.1) is required to hold globally, i.e. for the full Poincaré or Möbius group for all $A \in \mathcal{A}$, as our selection criterion for covariant sectors. However, the composition and conjugation rules (4.2 + 3) make sense only as long as $g_t(I)$ keep at space-like separation from ξ when g_t is a path in G of homotopy class $\tilde{g}$. The pre-image of the cocycle is then a local operator in $\mathcal{A}_\xi$. For larger $\tilde{g}$, $\mathcal{U}_{\sigma\rho}$ and $\mathcal{U}_{\bar{\rho}}$ can be computed as products of smaller transformations.

Let the space-like rotations resp. rigid rotations of the circle by the angle φ be implemented by $\mathcal{U}_\rho(\varphi)$. It follows from (4.1) that $\mathcal{U}_\rho(2\pi)$ commutes with the irreducible representation $\pi_0\rho(\mathcal{A})$, and therefore is a scalar:

$$\mathcal{U}_\rho(2\pi) = e^{2\pi i h_\rho} \,. \tag{5.2.9}$$

In the conformal case, we define the primary scaling dimension h_ρ to be the infimum of the spectrum of the generator L_0 of $\mathcal{U}_\rho(\varphi)$ in the sector $\mathcal{H}_\rho$. The infimum exists due to the spectrum condition. In the three-dimensional case, we choose the spin quantum number $0 \leq h_\rho < 2\pi$. Clearly, $\mathcal{U}_0(2\pi) = 1$ and $h_0 = 0$.

Now let ρ be localized in some interval (cone) I small enough such that I has no intersection with the rotated interval (cone) πI. We apply the argument leading to (5.2.2) with $J = \pi I$, $V = \mathcal{U}_0(\pi)^*\mathcal{U}_\rho(\pi)$ and $\hat{\rho} = \alpha_\pi \circ \rho \circ \alpha_\pi$. In the intertwiners of (5.2.2) we recognize the cocycles arising in (4.2): $\mathcal{A}_\zeta \ni Y_\rho(-\pi) = V_-^*$ and $\mathcal{A}_\xi \ni Y_\rho(+\pi) = e^{2\pi i h_\rho} V_+^*$. Therefore, we can compute for irreducible σ, ρ

$$\begin{aligned} \mathcal{U}_{\sigma\rho}(2\pi) &= \mathcal{U}_{\sigma\rho}(\pi)\mathcal{U}_{\sigma\rho}(-\pi)^* \\ &= \pi_0\sigma(e^{2\pi i h_\rho} V_+^*)\mathcal{U}_\sigma(\pi) \cdot [\pi_0\sigma(V_-^*)\mathcal{U}_\sigma(-\pi)]^* = e^{2\pi i h_\rho} e^{2\pi i h_\sigma} \pi_0\sigma(V_\rho)\,. \end{aligned} \tag{5.2.10}$$

Multiplying with $\pi_0(T)$ from the right, where $T : \tau \longrightarrow \sigma\rho$, τ irreducible, yields with (4.4), (5.2.3), and (3.7) $e^{2\pi i (h_\tau - h_\rho - h_\sigma)} = \kappa_\tau / \kappa_\sigma \kappa_\rho$. This is a weak version of the Spin-Statistics theorem [7, 21]:

Proposition 5.3. *Let ρ, σ, τ be irreducible DHR endomorphisms. If $[\tau]$ arises as a subsector of $[\sigma\rho]$, then*

$$e^{2\pi i (h_\tau - h_\sigma - h_\rho)} = \frac{\kappa_\tau}{\kappa_\sigma \kappa_\rho}\,, \quad \textit{in particular} \quad e^{2\pi i (h_\rho + h_{\bar{\rho}})} = \kappa_\rho^2\,. \tag{5.2.11}$$

The result of this subsection hold as well in the 2 + 1-dimensional situation. The point at infinity ξ has to be replaced by a space-like direction $r \in \mathbb{M}^3$, $r^2 = -1$, and $\mathcal{A}_\xi$ by $\mathcal{A}_r = \overline{\{\mathcal{A}(S) | S \not\ni r\}}$, where "$r \in S$" is understood in the asymptotic sense.

5.3. *The conformal exchange algebra on the circle*

The Proposition 5.3 implies $e^{2\pi i h_\rho} = \kappa_\rho$ provided there happens to be some irreducible sector $[\sigma]$ such that $[\sigma\rho]$ contains $[\sigma]$ as a subsector. In general, however, this stronger relation can be obtained only by combining algebraic with analytic properties, but the latter require some further physical input. In order to keep the analytic part which is beyond the scope of this article as short as possible, we shall concentrate in this subsection on the conformal light-cone case, where by scale invariance it is very reasonable to postulate the existence of well-behaved point-like limits. An analogous treatment should also be possible for the three-dimensional theory, but will not be done here.

In order to derive the strong version of the Spin-Statistics theorem in chiral conformal theories, it is sufficient to consider a subalgebra $\mathcal{A}_\xi$ w.r.t. some fixed point at infinity ξ. We may construct the corresponding reduced field bundle $\mathcal{F}_{\text{red}\xi} = \overline{\{F(e,\, A) | A \in \mathcal{A}_\xi\}}$ by choosing reference endomorphisms $\rho_\alpha \in \Delta_{\text{red}}$ of $\mathcal{A}_\xi$ localized in I, $\xi \notin I$, and local intertwiners $T_e \in \mathcal{A}_\xi$. The latter exist by Haag duality on the circle. The results of Sec. 2 and 3 hold within $\mathcal{A}_\xi$ resp. $\mathcal{F}_{\text{red}\xi}$. We shall therefore omit the label ξ and identify the uncompactified light-cone $S^1 \setminus \{\xi\}$ with $\mathbb{R}$. The "universal" reduced field bundle will be discussed in the end of this subsection.

Consider the 2-point function

$$f_\lambda(x-y) := \langle \alpha_x \alpha_\lambda(F(e,\, A))\Omega,\, \alpha_y \alpha_\lambda(F(e,\, A))\Omega \rangle\,,$$

where $F(e,\, A)$ is a localized operator with e of type $(0, \rho,\, \rho)$. α_λ is a scale transformation to be considered later in the point-like limit $\lambda \downarrow 0$, and α_x, α_y are translations of the uncompactified light-cone. Weak Locality (3.14) tells us

$$\langle F_2\Omega,\, F_1\Omega \rangle = \kappa_\rho^{-1} \langle \overline{F}_1\Omega,\, \overline{F}_2\Omega \rangle \quad \text{for } \mathcal{O}_2 > \mathcal{O}_1\,, \tag{5.3.1}$$

therefore for λ sufficiently small ($|x-y|$ sufficiently large)

$$f_\lambda(x-y) = \kappa_\rho^{\operatorname{sign}(y-x)} \bar{f}_\lambda(y-x)\,, \tag{5.3.2}$$

where $\bar{f}_\lambda$ is defined as f_λ with $\overline{F(e,\, A)}$ instead of $F(e,\, A)$.

We want to compare this behaviour with that of 2-point functions of conformal fields

$$\begin{aligned} f(x-y) &= \langle \varphi_d(x)\Omega,\, \varphi_d(y)\Omega \rangle = e^{-i\pi d}(x-y-i\varepsilon)^{-2d} \\ \bar{f}(y-x) &= \langle \bar{\varphi}_{\bar{d}}(y)\Omega,\, \bar{\varphi}_{\bar{d}}(x)\Omega \rangle = e^{-i\pi \bar{d}}(y-x-i\varepsilon)^{-2\bar{d}} \end{aligned} \tag{5.3.3}$$

understood as distributional boundary values $\varepsilon \downarrow 0$ determined by the spectrum condition (the orientation of the light-cone is chosen from its projection onto the time axis). The scaling dimensions d, $\bar{d}$ lie in the spectrum of L_0 (cf. (5.2.9)), i.e. $d = h_\rho + n$, $\bar{d} = h_{\bar{\rho}} + \bar{n}$. The coefficients in (5.3.3) are determined by positivity. These 2-point functions satisfy a relation of the same form as (5.3.2)

$$f(x-y) = e^{2\pi i d \cdot \operatorname{sign}(y-x)} \bar{f}(y-x) \quad \text{provided} \quad d = \bar{d}\,. \tag{5.3.4}$$

Since we expect that in the point-like limit $\lambda \downarrow 0$ the (appropriately rescaled) functions f_λ, $\bar{f}_\lambda$ tend to conformal 2-point functions of the form (5.3.3), we would conclude that the scaling dimensions of the limit fields of $\alpha_\lambda(F)$ and $\alpha_\lambda(\bar{F})$ coincide, and the phases in (5.3.2) and (5.3.4) also coincide. Therefore, $d = \bar{d} = h_\rho + n$ and $e^{2\pi i h_\rho} = e^{2\pi i h_{\bar\rho}} = \kappa_\rho$. For this conclusion it is necessary that the limit $\lambda \downarrow 0$ can be controlled. In particular, the localized operator $F = F(e, A)$ we started from must be chosen such that the high-energy spectral content of the vector $F\Omega \in \mathcal{H}_\rho$ is sufficiently well-behaved. For a detailed construction of a dense set of such vectors in the vacuum sector see [39]. We do not expect any essential complications for the charged sectors.

It is clear that in the point-like limit, $\alpha_\lambda(F)\Omega = \mathcal{U}_\rho(\lambda)F\Omega$ gets dominated by its contribution belonging to the Möbius group representation with the lowest scaling dimension (in general the primary one). It was also discussed in [39] that one can apply suitable (real) polynomials P_d of the Casimir operator of the Lie algebra of the Möbius group which annihilate the spectral contributions with scaling dimensions $< d$. Understood as polynomials in infinitesimal transformations $\alpha_{\tilde g}$, P_d are *local* operations on the reduced field bundle. Therefore one can as well obtain the higher (quasi-primary) fields as point-like limits of suitable localized operators from the reduced field bundle acting on the vacuum:

$$(\varphi_d^{(A)})e(x) := \lim_{\lambda\downarrow 0} \lambda^{-d} \cdot \alpha_x\alpha_\lambda \circ P_d(F(e, A)) . \tag{5.3.5}$$

Since $\alpha_{\tilde g}$ and thus P_d commute with the charge conjugation operation, and the latter also commute with conformal transformations, the same argument as before shows that the full spectra of scaling dimensions in the sectors ρ and $\bar\rho$ coincide. We summarize the discussion by

Proposition 5.4. *(Spin-Statistics Theorem) On the premises that the* on-vacuum *point-like limit (5.3.5) yields fields $(\varphi_d)_e(x)$ which generate a dense subspace of $\mathcal{H}_{\rm red}$ from the vacuum vector Ω, the spectrum of chiral scaling dimensions of fields of charge $[\rho]$ is contained in $h_\rho + \mathbb{N}_0$, with*

$$\mathcal{U}_\rho(2\pi) \equiv e^{2\pi i h_\rho} = \kappa_\rho . \tag{5.3.6}$$

The spectrum is charge conjugation invariant. In particular, $h_\rho = h_{\bar\rho}$.

By (4.5), the action of the conformal group on reduced field bundle operators is independent of the source and range sectors, and therefore the limit (5.3.5) is of the form $F(e, \lim \alpha_{\rho,\lambda}(A))$ with a *local* operation $\alpha_{\rho,\lambda}$ on the local degree of freedom induced by (4.5), depending only on the charge ρ. Thus, if the limit exists on the vacuum, by virtue of the commutation relations $F(e, \alpha_{\rho,\lambda}(A))F(e_1, B)\Omega = \sum R^{e\circ e_1}_{f_1\circ f} F(f_1, B)F(f, \alpha_{\rho,\lambda}(A))\Omega$ with $s(f) = s(e_1) = [0]$ it exists also on dense subspaces of the charged sectors. Therefore it is at least plausible that, in a suitable sense, (5.3.5) remains valid for e of general type. These objects will have all the expected properties of conformally covariant point-like vertex operators affiliated with the reduced field bundle. Since relative localizations as well as the various conjugation operations of Sec. 3 are preserved in the point-like limit, these vertex operators satisfy the exchange algebra [12] commutation relations (3.5) as well as the algebraic relations (3.9—15). We shall therefore refer to (5.3.5) as exchange fields.

We have not rigorously derived the existence of the limits (5.3.5). But this is what is expected at least in specific models, and we shall now restrict our further analysis to theories where the expressions (5.3.5) are well-defined off-vacuum (in the sense of distributions).

Definition 5.5. The algebra of unbounded fields $(\varphi_d)_e(x)$ obtained as *off-vacuum* point-like limits (5.3.5) is called the *exchange field algebra* $\mathcal{F}_{\rm exc}$ affiliated with the reduced field bundle $\mathcal{F}_{\rm red}$. We *assume* that the exchange fields with $s(e) = [0]$ generate a dense subspace of the Hilbert space from the vacuum, and that together with the latter also the exchange fields with arbitrary source and range are well-defined. The conjugation operations F^*, $\hat{F}$ and $\overline{F}$ apply in an obvious sense also to the exchange fields. By construction, the commutation relations (3.5) and conjugation properties (3.9–15), in particular Weak Locality, remain valid for exchange fields, and exchange fields transform under translations and dilatations in the canonical way dictated by the scaling dimension d.

Note that the dependence on the specific choice of the local degree of freedom A will be essentially lost in the point-like limit except for some possible degeneracy of quasi-primary fields with the same scaling dimension d. E.g., charge conjugation $\overline{(\varphi_d^{(A)})_e} = \sqrt{d_\alpha/d_\beta}(d_\rho/\chi_\rho)\Sigma_{\bar{e}}\zeta_{e\bar{e}} \cdot (\varphi_d^{(\bar{\rho}(A^*)R_\rho)})_{\bar{e}}$ goes along with a matrix between these internal spaces for conjugate charges, as a relic of the map $A \mapsto \bar{\rho}(A^*)R_\rho$.

Let us now turn to the CPT theorem for the reduced field bundle and the affiliated exchange field algebra. For this purpose we have to consider the space-time inversion $x \mapsto -x$. Note that on the light-cone, space and time inversion do not have a separate meaning. Since we are mainly interested in the role of the algebraic structures derived in Sec. 3 from the DHR theory of superselection sectors, we shall be rather short about the analytical analysis. One may proceed in complete parallelity with [34] (for local fields in chiral conformal quantum field theory) or even the standard treatment of the Wightman theory [31, 32].

In a first step one defines vectors $\varphi_n(w_n)\ldots\varphi_1(w_1)\Omega$ for complex variables with Im $w_1 > \ldots >$ Im $w_n > 0$ (the forward tube) by analytical continuation in the translation operators in $\varphi_n(x_n)\ldots\varphi_1(x_1)\Omega = e^{i(x_n-x_{n-1})P}\varphi_n(0)\ldots\varphi_1(0)e^{ix_1P}\Omega$. This is possible due to the spectrum condition.

In a second step one verifies (with the Edge-of-the-Wedge theorem) that in fact the real Jost points $x_n > \ldots > x_1$ resp. $x_n < \ldots < x_1$ lie in the *interior* of a larger domain of analyticity (the extended tube). For this argument, [34] use local commutativity, but the Weak Locality (Proposition 3.7) is all what is actually needed.

Next one exploits the extended analyticity and the covariance w.r.t. *real* scale transformations to show that complex scale transformations are well-defined on n-point vectors at appropriate points in the analyticity domain, with the expected results at the Jost points $0 < x_n < \ldots < x_1$ resp. $0 > x_n > \ldots > x_1$

$$\mathcal{V}(i\pi)\varphi_n(x_n)\dots\varphi_1(x_1)\Omega = \Pi_i((-1)^{n_i}\kappa_i^{\frac{1}{2}})\cdot\varphi_n(-x_n)\dots\varphi_1(-x_1)\Omega \quad \text{resp.}$$
$$\mathcal{V}(-i\pi)\varphi_n(x_n)\dots\varphi_1(x_1)\Omega = \Pi_i((-1)^{n_i}\kappa_i^{-\frac{1}{2}})\cdot\varphi_n(-x_n)\dots\varphi_1(-x_1)\Omega\,. \tag{5.3.7}$$

Here $\mathcal{V}(t) = e^{itD}$ implement the conformal dilatations. $\varphi_i = (\varphi_{d_i})_{e_i}$ have scaling dimensions $d_i = h_{\rho_i} + n_i$ giving rise to the transformation factors $e^{i\pi d} = (-1)^n\kappa_\rho^{1/2}$. From now on we fix our choice $\kappa_\rho^{1/2} = e^{i\pi h_\rho}$ which was free until now (cf. A.1). Then we have

Proposition 5.6. (*CPT Theorem*) *There is an anti-unitary operator* Θ *on* $\mathcal{H}_{\mathrm{red}}$ *which implements the CPT symmetry in the exchange field algebra:*

$$\Theta\cdot(\varphi_d^{(A)})_e(x)\cdot\Theta^{-1} = (-1)^n\cdot\overline{(\varphi_d^{(A)})_e}(-x)\,. \tag{5.3.8}$$

$\overline{(\varphi_d^{(A)})_e}$ *carries the conjugate charge and interpolates between the conjugate sectors.* Θ *leaves the vacuum invariant, and commutes CPT-covariantly with translations and dilatations. It is an involution up to a phase:*

$$\Theta^2|_{\mathcal{H}_\rho} = \chi_\rho\,. \tag{5.3.9}$$

Proof. One variant of the proof (of which we omit all details) proceeds in analogy to the reasoning in [31, 32]: consider an n-point function at $x_1 > \dots > x_n$. By translation invariance, we may even assume $x_n > 0$. Then (in evident short-hand notation)

$$\begin{aligned}\langle\Omega,\,\varphi_n(x_n)\dots\varphi_1(x_1)\Omega\rangle^* &= \langle\Omega,\,\varphi_1(x_1)^*\dots\varphi_n(x_n)^*\Omega\rangle\\ &= \Pi_i\kappa_i^{-\frac{1}{2}}\langle\Omega,\,\widehat{\varphi_n^*}(x_n)\dots\widehat{\varphi_1^*}(x_1)\Omega\rangle\\ &= \Pi_i\kappa_i^{-\frac{1}{2}}\langle\Omega,\,\overline{\varphi_n}(x_n)\dots\overline{\varphi_1}(x_1)\Omega\rangle\\ &= \Pi_i(-1)^{n_i}\langle\Omega,\,\overline{\varphi_n}(-x_n)\dots\overline{\varphi_1}(-x_1)\Omega\,.\end{aligned}$$

Here, the second equality is the Weak Locality 3.7., the third equality is the definition (3.12), and the last equality follows from (5.3.7). Therefore, the replacement $\varphi(x)\mapsto(-1)^n\bar\varphi(-x)$ leads to a complex conjugation of all n-point functions at ordered points in configuration space. The commutation relations (3.5) together with the symmetry of R-matrices (A.5(e)) imply the same relation for arbitrary configurations. Thus, defining the CPT operator on multi-field vectors in $\mathcal{H}_\rho$ by

$$\Theta\varphi_m(x_m)\dots\varphi_1(x_1)\Omega := \Pi_i(-1)^{n_i}\cdot\overline{\varphi_m}(-x_m)\dots\overline{\varphi_1}(-x_1)\Omega \tag{5.3.10}$$

(where $r(e_m) = [\rho]$) consistently yields an anti-unitary operator, and by (3.13), Θ^2 on such a state is given by $\Pi_i\chi(r(e_i))/\chi(s(e_i)) = \chi_\rho$.

We want to present an alternative proof which is closer to the Tomita-Takesaki modular theory. It is inspired by [33] and parallels the more detailed elaboration for the vacuum sector in [34, 39].

Define the antilinear operator Θ on a dense subset of $\mathcal{H}_\rho$ by

$$\Theta : (\varphi_d^{(A)})_e(x)\Omega \mapsto (-1)^n \cdot \overline{(\varphi_d^{(A)})_e(-x)}\Omega .$$

Clearly, Θ commutes CPT-covariantly with translations and dilatations, and by the form (5.3.3) of conformal 2-point functions and (the point-like limit of) the Weak Locality relation (5.3.1)

$$\begin{aligned}\langle\Theta\varphi_1(x)\Omega,\, \Theta\varphi_2(y)\Omega\rangle &= \langle\overline{\varphi_1}(-x)\Omega,\, \overline{\varphi_2}(-y)\Omega\rangle \\ &= \kappa_\rho^{\mathrm{sign}(x-y)} \cdot \langle\overline{\varphi_1}(x)\Omega,\, \overline{\varphi_2}(y)\Omega\rangle \\ &= \langle\varphi_2(y)\Omega,\, \varphi_1(x)\Omega\rangle .\end{aligned}$$

Therefore, Θ is anti-unitary on a dense subspace and can be uniquely continued to an anti-unitary operator on $\mathcal{H}_{\mathrm{red}}$. The point-like limit of (3.13) implies (5.3.9). Now, let us relate Θ to a modular conjugation. Let $x > 0$ and $F \in \mathcal{F}_{\mathrm{red}}(\mathbb{R}_+)$ a suitably regularized operator [34] such that $\mathcal{V}(t)F\Omega$ has an analytic continuation in t. Then

$$\begin{aligned}\langle\mathcal{V}(i\pi)F\Omega,\, \varphi(x)\Omega\rangle &= \langle F\Omega,\, \mathcal{V}(i\pi)\varphi(x)\Omega\rangle = (-1)^n\kappa_\rho^{\frac{1}{2}}\langle F\Omega,\, \varphi(-x)\Omega\rangle \\ &= (-1)^n\kappa_\rho^{-\frac{1}{2}}\langle\bar\varphi(-x)\Omega,\, \overline{F}\Omega\rangle = \kappa_\rho^{-\frac{1}{2}}\langle\Theta\varphi(x)\Omega,\, \overline{F}\Omega\rangle \\ &= \kappa_\rho^{-\frac{1}{2}}\langle\Theta^*\overline{F}\Omega,\, \varphi(x)\Omega\rangle ,\end{aligned}$$

where we have used again Weak Locality (5.3.1). As in [34], we may conclude (with the operator $\kappa^{1/2}$ defined by its eigenvalues $\kappa_\rho^{1/2}$ on $\mathcal{H}_\rho$)

$$\mathcal{V}(i\pi)F\Omega = \kappa^{\frac{1}{2}}\Theta^*\overline{F}\Omega \quad \text{for} \quad F \in \mathcal{F}_{\mathrm{red}}(\mathbb{R}_+) .$$

Now, consider the antilinear unbounded operators

$$S_\pm : F(e,\, A)\Omega \mapsto \overline{F(e,\, A)}\,\Omega \equiv \sqrt{d_\rho}/\chi_\rho \cdot F(\bar e,\, \bar\rho(A^*)R_\rho)\Omega \quad \text{for} \quad F \in \mathcal{F}_{\mathrm{red}}(\mathbb{R}_\pm) \tag{5.3.11}$$

defined on dense subspaces by the Reeh-Schlieder theorem (recall from (1.4) that $\mathcal{F}_{\mathrm{red}}(\mathcal{O})\Omega = \oplus_\alpha(\alpha,\, \pi_0(U_\alpha^*) \cdot \pi_0(\mathcal{A}(\mathcal{O}))\Omega)$ with suitable unitary charge transporters U_α, therefore Ω is cyclic and separating for the sets $\{F(e,\, A) \in \mathcal{F}_{\mathrm{red}}(\mathbb{R}_\pm) | s(e) = [0]\}$ if the same holds for $\mathcal{A}(\mathbb{R}_\pm)$ in the vacuum sector). The Weak Locality (5.3.1) shows that S_-^* is defined on the domain of S_+ where it coincides up to a factor with S_+, and vice versa:

$$\kappa^{-1}S_+ \subset S_-^*, \quad \kappa^{+1}S_- \subset S_+^* .$$

Having densely defined adjoints, $S_\pm$ are closable. The domains of definition (5.3.11) are cores for the operators $\mathcal{V}(\pm i\pi)$. Therefore, as in [34], the preceding calculation shows

Proposition 5.7. *Let $S_\pm$ be the closures of the operators (5.3.11). Then their polar decompositions $S = J \cdot \Delta^{1/2}$ are*

$$S_+ = \kappa^{\frac{1}{2}}\Theta \cdot \mathcal{V}(i\pi), \quad S_- = \kappa^{-\frac{1}{2}}\Theta \cdot \mathcal{V}(-i\pi) . \tag{5.3.12}$$

The modular conjugations Ad_J map the exchange field algebras $\mathcal{F}_{\mathrm{exc}}(\mathbb{R}_\pm)$ associated with the half-lines into each other, and the modular transformations $\mathrm{Ad}_{\Delta^{it}}$ are the real scale transformations leaving $\mathcal{F}_{\mathrm{exc}}(\mathbb{R}_\pm)$ invariant.

It remains to show (5.3.8). For this purpose we compute for, e.g., $x_1 > x_2 > 0$

$$\begin{aligned}\Theta\varphi_2(x_2)\varphi_1(x_1)\Omega &= \kappa_\rho^{\frac12} S_- \mathcal{V}(i\pi)\varphi_2(x_2)\varphi_1(x_1)\Omega \\ &= \kappa_\rho^{\frac12}(-1)^{n_1+n_2}(\kappa_1\kappa_2)^{-\frac12}\cdot\overline{\varphi_2(-x_2)\varphi_1(-x_1)}\Omega \\ &= (-1)^{n_1+n_2}\overline{\varphi_1}(-x_2)\overline{\varphi_1}(-x_1)\Omega \\ &= (-1)^{n_1+n_2}\overline{\varphi_2}(-x_2)\kappa_1^{-\frac12}\Theta\mathcal{V}(-i\pi)\varphi_1(-x_1)\Omega \\ &= (-1)^{n_2}\overline{\varphi_2}(-x_2)\Theta\varphi_1(x_1)\Omega\,,\end{aligned}$$

where we have used Proposition 3.8 in the third equality. The same result holds for $x_1 < x_2 < 0$, and by translation covariance for $x_1 \neq x_2$. By comparison, we get (5.3.8).

The Proposition 5.7 proved *en passant* shows that the charge conjugation operation $F \mapsto \overline{F}$ gives rise to a generalized Tomita-Takesaki theory, with a direct geometric interpretation of the modular structure. In fact, the operators $S_\pm$ may be identified with well-known *relative* Tomita-Takesaki conjugations of the *observable algebra* w.r.t. suitable pairs of vectors in $\mathcal{H}_\rho$ and $\mathcal{H}_{\bar\rho}$. The new result is their action on the interpolating exchange fields. It is tempting to ask whether such a theory can be directly established in a generalized von Neumann algebraic setting, based on a "twisted * conjugation" (the present operation $F \mapsto \overline{F}$ satisfying (3.13 + 16)) and giving rise to the "twisted commutant" without resorting to the underlying algebra of observables, locality, and exchange fields.

Let us now turn to the "universal" reduced field bundle $\mathcal{F}_{\mathrm{red,univ}}$ on the circle (resp. in 2 + 1 dimensions). It is again defined by (1.2) but with $A \in \mathcal{A}_{\mathrm{univ}}$, while for definiteness, all $\rho_\alpha \in \Delta_{\mathrm{red}}$ should be chosen localized in the same proper interval I and the bases $\{T_e\}$ are *local* intertwiners in $\mathcal{A}(I)$.

Due to the existence of *global* intertwiners, the notion (1.4) of "localization" is no longer equivalent to (1.3) in $\mathcal{F}_{\mathrm{red,univ}}$. Moreover, since there is no global left/right distinction, the status of the commutation relations (3.5) must be re-examined. (This observation does not invalidate our derivation of Proposition 5.4–5.7, since although we have exploited the covariance (4.1) of the endomorphisms under the full Möbius group, the reduced field bundle was only used w.r.t. some fixed point at infinity, where the commutation relations (3.5) and their consequences (3.14 + 15) hold in the naive sense). In fact, the discussion in Sec. 5.1 and 5.2 suggests that the correct notion of localization should be linked to the specific charge transporter $U : \rho \longrightarrow \mathrm{Ad}_U \circ \rho$ which satisfies (1.4), leading to localization in a covering space.

By (4.5), the space-time covariance is implemented in $\mathcal{F}_{\mathrm{red,univ}}$. $F_{\mathrm{red,univ}}$ contains all $\mathcal{F}_{\mathrm{red}\xi}$ ($\xi \not\ni I$) as subalgebras. These are stable only under conformal transformations leaving ξ invariant. Yet, elementwise the conformal covariance is implemented for a neighbourhood of unity in the (covering of the) Möbius group, as long as the transformed localization intervals do not move across ξ.

In order to understand the action of the full Möbius group, and to convince ourselves that the left/right ambiguity w.r.t. different $\xi \in S^1$ does not lead to any contradictions on the circle between commutation relations (3.5) valid in either $\mathcal{F}_{\mathrm{red}\xi}$, let us study again the rigid rotations of S^1. For instance, applying α_π twice according to (4.5) yields

$$\alpha_{2\pi}(F(e, A)) = F(e, Y_\rho \cdot A) \quad \text{with} \quad Y_\rho := Y_\rho(\pi)\alpha_\pi(Y_\rho(\pi)) . \tag{5.3.13}$$

As it stands, this operator is in general not contained in any $\mathcal{F}_{\mathrm{red}\xi}$, since Y_ρ is a global element of $\mathcal{A}$ not contained in any local algebra. But (5.3.13) can be identified with a localized reduced field bundle operator as follows. Note as before (with I, $J = \pi I$, ξ, ζ and $V_\pm$ as in Sec. 5.2) that $Y_\rho(\pi) = e^{2\pi i h_\rho} V_+^*$, and $\alpha_\pi(Y_\rho(\pi)) = V_-$. Therefore, the global self-intertwiner Y_ρ in (5.3.13) equals

$$Y_\rho = \kappa_\rho \cdot V_\rho .$$

Since by (5.2.3) V_ρ is given in the representation $\pi_0\rho_\alpha$ by the monodromy, and T_e diagonalize the monodromy (Lemma 3.3), the action (1.2) of $\alpha_{2\pi}(F(e, A))$ on $\mathcal{H}_{\mathrm{red}}$ equals (e of type (α, ρ, β))

$$\alpha_{2\pi}(F(e, A)) = (\kappa_\beta/\kappa_\alpha) \cdot F(e, A) \tag{5.3.14}$$

in accordance with the definition of $\alpha_{\tilde{g}}$: $\mathrm{Ad}_{\mathcal{U}(\tilde{g})}$ and the values of $\mathcal{U}(2\pi)$ in the source and range spaces. This is of course the periodicity expected for conformal exchange fields [12]. Now let $F_1 \in \mathcal{F}_{\mathrm{red}\xi}(I)$ and $F_2 \in \mathcal{F}_{\mathrm{red}\xi}(\pi I)$. Then $F_1 \in \mathcal{F}_{\mathrm{red}\zeta}(I)$ and $F_2 = \alpha_{2\pi}(G_2)$ where $G_2 \in \mathcal{F}_{\mathrm{red}\zeta}(\pi I)$. Since w.r.t. ξ, πI lies to the right of I, the commutation relation (3.5) between F_2 and F_1 within $\mathcal{F}_{\mathrm{red}\xi}$ involves $R(+)$, and similarly the commutation relation between G_2 and F_1 within $\mathcal{F}_{\mathrm{red}\zeta}$ involves $R(-)$. In view of the periodicity (5.3.14), these two relations are compatible iff (3.6) holds: in fact, this type of argument for exchange fields originally led to Proposition 3.2 as a restriction on the R-matrices of conformal exchange algebras in [12], see the remark after Proposition 3.2.

By the preceding calculations, we are led to the following definition. Choose an interval $J_0 \subset \mathbb{R}$ (of extension $< 2\pi$) of the universal covering $\mathbb{R}$ of S^1 which projects onto $I \subset S^1$.

Definition 5.8. Let $J \subset \mathbb{R}$ be an interval (of extension $< 2\pi$) of the universal covering $\mathbb{R}$ of S^1. For $\rho \in \Delta_{\mathrm{red}}$ the pair (ρ, A) is said to be localized in J if there is a local operator $C \in \mathcal{A}(I)$ such that all $F(e, A)$ with charge ρ are the translates of $F(e, C) \in \mathcal{F}_{\mathrm{red}\xi}(I)$ under $\alpha_{\tilde{g}}$, where $\tilde{g}(J_0) = J$.

In other words, A is obtained from C by a sequence of "small" transformations as in (4.5) which sum up to $\tilde{g}$. This notion of localization is consistent with (1.3) in every $\mathcal{F}_{\mathrm{red}\xi}$. It does not refer to the individual operators $F(e, A) \in \mathcal{F}_{\mathrm{red,univ}}$ for which a source- and range-dependent periodicity holds, but to the *collection* of such operators with common (ρ, A). (These are essentially the elements of the original field bundle [4].) The covariance (4.5) lifts to these pairs in the obvious way. The commutation relations (3.5) are then globally valid in the form

Proposition 5.9. *Let (ρ_1, A_1) and (ρ_2, A_2) be localized in intervals J_1 and J_2 which project onto disjoint intervals on the circle. Determine $N \in \mathbb{N}$ such that $J_1 + 2\pi N < J_2 < J_1 + 2\pi(N+1)$. Then*

$$F(e_2, A_2) \cdot F(e_1, A_1) = \sum_{f_1 \circ f_2} R^{e_2 \circ e_1}_{f_1 \circ f_2}(N) \cdot F(f_1, A_1) \cdot F(f_2, A_2), \qquad (5.3.15)$$

where $R^{e_2 \circ e_1}_{f_1 \circ f_2}(N) = \pi_0[T^*_{e_2} T^*_{e_1} \rho_\alpha(\varepsilon_N(\rho_2, \rho_1)) T_{f_2} T_{f_1}]$ are matrix elements in the vacuum representation of the generalized statistics operators

$$\varepsilon_N(\rho_2, \rho_1) = \rho_1(Y^N_{\rho_2}) \varepsilon(\rho_2, \rho_1) Y^{-N}_{\rho_2} : \rho_2\rho_1 \to \rho_1\rho_2 .$$

One has (notation as in Proposition 3.2)

$$R^{e_2 \circ e_1}_{f_1 \circ f_2}(N) = \left(\frac{\kappa_\alpha \kappa_\gamma}{\kappa_\beta \kappa_\delta}\right)^N \cdot R^{e_2 \circ e_1}_{f_1 \circ f_2}(+) . \qquad (5.3.16)$$

Clearly, $N = 0$ and $N = -1$ correspond to the ordinary statistics operator and its opposite, with matrices $R(+)$ and $R(-)$. The other ε_N are in general global operators. Therefore the above self-intertwiners $T^*_{e_2} \dots T_{f_1} : \rho_\gamma \to \rho_\gamma$ will be scalars only if evaluated in π_0. This proposition can be directly proven, if one transports both F_i by suitable powers of $\alpha_{2\pi}$ into some $\mathcal{F}_{\mathrm{red}\xi}$ and controls these translations by the formula (5.3.13). A common shift by $\alpha_{2\pi M}$ turns out immaterial by the identity following from (5.2.3) (but not from (2.2) which does not apply to global intertwiners!)

$$\rho_1(Y_2) Y_1 \varepsilon(\rho_2, \rho_1) = \varepsilon(\rho_2, \rho_1) Y_2 \rho_2(Y_1) .$$

Note that in view of (5.2.3), the evaluation of ε_N in $\pi_0\rho_\alpha$ gives the representative of a complicated braid $(\sigma_1\sigma_0^2\sigma_1)^N \sigma_1 (\sigma_0^2)^{-N}$ involving an extra generator σ_0 corresponding to a "zero'th" string for the source sector.

The geometric reason for this observation is that the unitary operators $\rho^{i-1}(\varepsilon(\rho, \rho))(i = 1 \dots n-1)$ and $V_\rho = \kappa_\rho^{-1} Y_\rho$ in fact represent the braid group of the cylinder, as one should expect for a theory over the circle. Namely, they satisfy the relations of σ_i as in B_n and τ : $\tau\sigma_i = \sigma_i\tau (i \neq 1)$ and $\tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau$. (Proof: $\varepsilon(\rho, \rho) V_\rho \varepsilon(\rho, \rho) = \rho(V_\rho)$, and V_ρ commutes with $\rho(\mathcal{A})$.) Putting $\zeta = \sigma_{n-1} \dots \sigma_1 \tau$, these relations translate into $\sigma_i \zeta = \zeta \sigma_{i+1}$ $(i = 1 \dots n-2)$ and $\zeta^2 \sigma_1 = \sigma_{n-1} \zeta^2$, the defining relations for the braid group of the cylinder. On the other hand, the above relations of τ are those of σ_0^2 within B_{n+1}. Therefore the subgroup of B_{n+1}

generated by σ_i ($i = 1 \ldots n-1$) and σ_0^2 coincides with the braid group of the cylinder. These remarks generalize to the colored groupoids (for different charges involved) without difficulty.

By the Definition 5.8, a pair (ρ, A) may be localized in several intervals with the same projections onto the circle, whenever some power of Y_ρ is trivial. The spectrum of Y_ρ in the representation $\pi_0\rho_\alpha$ is exhausted by $\kappa_\beta/\kappa_\alpha$, $[\rho_\beta]$ contained in $[\rho_\alpha\rho]$. Therefore, if $(\kappa_\beta/\kappa_\alpha)^n = 1$ for all sectors α, β interpolated by ρ, then the fields of charge ρ rather live on an n-fold covering of S^1; in particular observables live on the original circle since $Y_0 = 1$. The commutation relations (5.3.15) remain unambiguous since (5.3.16) have the same periodicity.

6. Conclusions

In spite of its somewhat uncomfortable appearance, the reduced field bundle exhibits all the algebraic structures necessary for a decent quantum field theory of charged fields. At present, it is the only algebra available to describe charged fields with non-abelian braid group statistics in the generic case.

Prominent among these algebraic structures are a charge conjugation operation and the property of Weak Locality. They are derived from the DHR theory of super selection sectors with finite statistics, and illustrate the power of this approach. If space-time covariance is explicitly postulated, one may derive the Spin-Statistics theorem up to a sign, and if in addition affiliated point-like fields (the appropriate starting point of a Wightman axiomatic approach) exist on- and off-vacuum, the true Spin-Statistics theorem and the CPT theorem for charged fields follow along the usual lines. The often ignored remark in the textbooks that for CPT only a genuinely weaker requirement than locality is actually needed, deploys its full significance in the present context.

As mentioned in the introduction, the reduced field bundle also exists as a less familiar but physically equivalent description of four-dimensional theories with Fermi and Bose fields transforming under a global gauge group responsible for the super selection sectors, e.g., in the form of parafields [28]. It can be more directly obtained with the methods and results of the Doplicher-Roberts reconstruction [24], where it arises as the gauge invariant remnant after the Clebsch-Gordan coefficients for products of irreducible tensor operators have been removed. The structure constants D and R then coincide exactly with the $6j$-symbols of the gauge group.

This observation is at the basis of the hope to identify the quantum symmetry responsible for superselection sectors with braid group statistics by the study of the reduced field bundle. Although we did not discuss the issue of quantum symmetry, we hit upon its footprint again and again — in the guise of numerical identities among structure constants or of remarkable invariances of Markov traces giving rise to topological invariants of 3-manifolds, and most prominently in the structure of the center of the universal algebra and its representations. While some relation to quantum groups is undeniable, we insist in the problems which arise when quantum groups are to be implemented as symmetries of operator algebras acting in Hilbert space.

As in local quantum field theory, the CPT theorem for charged fields is closely related to the Tomita-Takesaki theory of modular conjugations. We have identified the appropriate conjugation operation, the polar decomposition of which yields — in a particularly simple kinematic situation — the anti-unitary CPT operator as well as a geometric group of modular automorphisms. While the Tomita-Takesaki theory has proved to be an extremely powerful tool in von Neumann algebra theory in terms of extremely simple algebraic structures (the operator adjoint and the commutant), the present "twisted" generalization refers largely to the rather involved superselection structure constants. Therefore, the understanding of the underlying abstract von Neumann algebra theory seems intimately linked to the understanding of braided categories.

The question to what extent the general structures analyzed in this paper are realized in models was not discussed. Certainly our *results* match the general findings in models of chiral conformal field theory. A full identification requires a lot of work and at the moment has led only to a partial confirmation. For various models of abelian [5] and non-abelian [41] current algebras, the axioms of the operator algebraic approach have been verified and the localized automorphisms leading to *simple* sectors (statistical dimension = 1) were found. The main limitation at present is the difficulty to write down localized endomorphisms for non-simple sectors. Here, some progress is made by considering the observables as a subalgebra of some larger algebra (e.g., the chiral Ising model expressed in terms of the Majorana algebra [27] and generalizations thereof [40], or "conformal embeddings" of higher-level non-abelian current algebras into those of level 1 [41]), and to study automorphisms of the latter in restriction to the former. More generally, it seems promising to generalize the theory to *positive maps* [42] which are much easier available than endomorphisms.

The "Casimir operators" corresponding to non-simple sectors which are predicted in Sec. 5.2 were not known before even in completely solved models like the minimal models [13]. It is an interesting test to try and find these central elements also in the conventional approach. This should also lead to a more *intrinsic* understanding of the underlying character structure for the chiral theory (i.e. not referring to the extrinsic two-dimensional principle of $SL(2, \mathbb{Z})$ modular invariance), and explain why it is to be expected also in three-dimensional theories with "string-like" charges.

Appendix

A. Structure Constant Symmetries

The numerical square matrices η_{ee^*} and $\theta_e^{\hat{e}}$ occurring in Sec. 3 are "coupling constants" for the charge conjugation structure of the theory. They may be regarded as matrix elements of linear and anti-linear maps between different intertwiner spaces which are related via charge conjugation and by (2.11) have the same dimensions. They depend covariantly on the choice of orthonormal bases $\{T_e\}$ (as indicated by the index position) and R_α. For $N_{\alpha\beta}^\gamma = 1$, the phases of the intertwiners may always be chosen such that $\theta = \pm 1$ and $\eta = \pm\sqrt{d_\gamma / d_\alpha d_\beta}$ with intrinsic signs for pseudoreal sectors involved. Some of the corresponding symmetries A.2 and A.3 have also been obtained in the algebraic setting by Fröhlich *et al.* [7].[4] We give here a list of relations which are helpful for algebraic transformations in the reduced field bundle, as well as related symmetries of R- (and D-)matrices. These relations are more or less easily derived from the fundamental identities (2.2–4), their consequences (2.6), (3.7 + 8), and the orthonormal choice (3.1). Some proofs can be found in the literature [20,43].

As an illustration, we present the algebraic background underlying the matrix η_{ee^*}. There are similar structures for $\theta^e_{\hat{e}}$ and $\zeta_{e\bar{e}}$. Consider the linear mapping η from the intertwiners: $\rho_\gamma \to \rho_\alpha\rho_\beta$ to the intertwiners: $\rho_\gamma\bar{\rho}_\beta \to \rho_\alpha$:

$$T_e \mapsto \rho_\alpha(R^*_{\bar{\beta}})T_e =: \sum_{e^*} \eta_{ee^*} T^*_{e^*} \quad \Longleftrightarrow \quad \eta_{ee^*} := \rho_\alpha(R^*_{\bar{\beta}})T_e T_{e^*}\,. \tag{A.1}$$

It induces an anti-linear mapping between the spaces spanned by T_e and T_{e^*}, with matrix elements η_{ee^*}. The unitarity of the square matrix η_{ee^*} up to the factor $d_\gamma/d_\alpha d_\beta$ (see (A.4(a)) below) is just the expression of the algebraic orthogonality for e, f with fixed common charge and range, but free source:

$$\rho_\alpha(R^*_{\bar{\beta}})T_e T^*_f\, \rho_\alpha(R_{\bar{\beta}}) = (d_\beta/d_\alpha d_\rho)\delta_{ef}\,. \tag{A.2}$$

This relation follows from, and is actually equivalent to (3.8), since applying ϕ_α and then omitting $\rho_\alpha(R^*)\ldots\rho_\alpha(R)$ are trivial operations with scalars. The symmetry (A.4(b)) reflects the involutivity $\eta(\eta(T)^*)^* = (1/d_\beta\chi_\beta)\cdot T$.

Definition A.1. For e of type $(\alpha,\,\rho,\,\beta)$, let e^* be of type $(\beta,\,\bar{\rho},\,\alpha)$, $\hat{e}$ of type $(\bar{\beta},\,\rho,\,\bar{\alpha})$, and $\bar{e}$ of type $(\bar{\alpha},\,\bar{\rho},\,\bar{\beta})$. Define (with some fixed choice of the square roots of the statistics phases).

$$\begin{aligned}\eta_{ee^*} &:= \rho_\alpha(R^*_{\bar{\beta}})T_e T_{e^*}\\ \theta^e_{\hat{e}} &:= \sqrt{\kappa_\rho\kappa_\beta/\kappa_\alpha}\cdot R^*_{\bar{\beta}}T^*_e\,\rho_\alpha(\varepsilon(\rho_{\bar{\beta}},\,\rho)T_{\hat{e}})R_{\bar{\alpha}}\\ &\equiv \sqrt{\kappa_\alpha/\kappa_\beta\kappa_\rho}\cdot R^*_{\bar{\beta}}T^*_e\,\rho_\alpha(\varepsilon(\rho,\,\rho_{\bar{\beta}})^*T_{\hat{e}})R_{\bar{\alpha}}\\ \zeta_{ee^*} := \Sigma_{e^*}\theta^{e^*}_{\bar{e}}\,\eta_{ee^*} &\equiv \sqrt{\kappa_\rho\kappa_\alpha/\kappa_\beta}\cdot R^*_{\bar{\rho}}R^*_{\bar{\alpha}}\rho_\alpha(\varepsilon(\rho_{\bar{\alpha}},\,\rho)^*)T_e\rho_\beta(T_{\bar{e}})R_{\bar{\beta}}\,.\end{aligned} \tag{A.3}$$

The two definitions of θ are equivalent in view of Lemma 3.3.

Lemma A.2. *The following identities hold (summation over repeated indices; e of type $(\alpha,\,\rho,\,\beta)$):*

$$\text{(a)}\;\; \eta_{ee^*}(\eta_{fe^*})^* = (d_\beta/d_\alpha d_\rho)\delta_{ef}$$
$$\text{(b)}\;\; \eta_{ee^*} = \chi_\rho(d_\beta/d_\alpha)\cdot\eta_{e^*e}$$

[4]It is stated in [7c, Theorem 5.8] that χ_ρ can always be put to $+1$ by adjusting the phase of $R_{\bar{\rho}}$: id $\to \rho\bar{\rho}$ relative to that of R_ρ : id $\to \bar{\rho}\rho$. But for self-conjugate sectors, there is no such freedom since $R_\rho = R_{\bar{\rho}}$. Thus the value of χ_ρ is intrinsic and may be -1 (pseudo-real sectors). Consequently, in the presence of pseudo-real sectors, one cannot have all η-matrices simultaneously positive diagonal as stated in [7c, Theorem 5.16 (ii)] (cf. (A.4(b)) below).

$$
\begin{aligned}
&\text{(c)}\ \theta^e_{\hat{e}}(\theta^f_{\hat{e}})^* = \delta_{ef} \\
&\text{(d)}\ \theta^e_{\hat{e}} = (\chi_\beta/\chi_\alpha)\cdot(\theta^{\hat{e}}_e)^* \\
&\text{(e)}\ \zeta_{e\bar{e}}(\zeta_{e\bar{f}})^* = (d_\beta/d_\alpha d_\rho)\delta_{ef} \\
&\text{(f)}\ \zeta_{e\bar{e}} = (\chi_\rho\chi_\alpha/\chi_\beta)\cdot\zeta_{\bar{e}e}\,.
\end{aligned}
\tag{A.4}
$$

Lemma A.3. *There are symmetries of the R- and D-matrices.*

$$
\begin{aligned}
&\text{(a)}\ \eta_{e_1^*e_1}\cdot R^{e_2\circ e_1}_{f_1\circ f_2}(\pm) = R^{f_1^*\circ e_2}_{f_2\circ e_1^*}(\mp)\cdot\eta_{f_1^*f_1} \\
&\text{(b)}\ \eta_{e_1^*e_1}\eta_{e_2^*e_2}\cdot R^{e_2\circ e_1}_{f_1\circ f_2}(\pm) = R^{f_2^*\circ f_1^*}_{e_1^*\circ e_2^*}(\pm)\cdot\eta_{f_1^*f_1}\eta_{f_2^*f_2} \\
&\text{(c)}\ \eta_{e_2e_2^*}\cdot D^{e_2\circ e_1}_{f,e} = [D^{e_2^*\circ e}_{f^*,e_1}]^*\cdot\eta_{ff^*} \\
&\text{(d)}\ \theta^{e_1}_{\hat{e}_1}\theta^{e_2}_{\hat{e}_2}\cdot R^{\hat{e}_1\circ\hat{e}_2}_{\hat{f}_2\circ\hat{f}_1}(\pm) = R^{e_2\circ e_1}_{f_1\circ f_2}(\pm)\cdot\theta^{f_1}_{\hat{f}_1}\theta^{f_2}_{\hat{f}_2} \\
&\text{(e)}\ \zeta_{e_1\bar{e}_1}\zeta_{e_2\bar{e}_2}\cdot R^{\bar{e}_2\circ\bar{e}_1}_{\bar{f}_1\circ\bar{f}_2}(\pm) = [R^{e_2\circ e_1}_{f_1\circ f_2}(\mp)]^*\cdot\zeta_{f_1\bar{f}_1}\zeta_{f_2\bar{f}_2} \\
&\text{(f)}\ \zeta_{e_1\bar{e}_1}\zeta_{e_2\bar{e}_2}\cdot D^{\bar{e}_2\circ\bar{e}_1}_{\bar{f},\bar{e}} = [D^{e_2\circ e_1}_{f,e}]^*\cdot\zeta_{f\bar{f}}\zeta_{e\bar{e}}\,.
\end{aligned}
\tag{A.5}
$$

We omit the proof of these symmetries from the coherence equations (2.2–4) and (3.1). For example, for (A.5(a)) apply ρ_β to the identity $R_1^*\bar{\rho}_1(\varepsilon(\rho_2,\rho_1)) = \rho_2(R_1^*)$ $\varepsilon(\rho_2,\bar{\rho}_1)^*$ and take matrix elements in path space as in Sec. 3 for the polynomial identities. Similarly, (c) is obtained from the defining relation $\rho_1(R_2^*)T_f = \eta_{ff^*}\cdot T_{f^*}^*$. For (e,f) see [43]. We omit further symmetries related to $e\mapsto\hat{e}$ of type (ρ,α,β) which are not relevant for the reduced field bundle.

The symmetries A.3. are not necessary for proofs in the present article, but arise *a posteriori* as consistency relations among the exchange algebra and the adjoint and conjugation structures. In particular, (A.5(e,f)) are the CPT symmetry (charge conjugation, space-time inversion, and complex conjugation) pre-existing in the DHR theory at the level of structure constants, without assuming covariance.

B. Invariants of 3-Manifolds

In [3] we have described the construction of link invariants from irreducible DHR endomorphisms. The representative $\varepsilon_\rho(b)$ of a braid $b\in B_n$ in terms of statistics operators $\varepsilon(\rho,\rho)$ is a self-intertwiner: $\rho^n\longrightarrow\rho^n$. Since the iterated left-inverse ϕ^n is a positive normalized trace on the commutant of ρ^n, the scalar function $\mathrm{tr}_\rho = \phi^n\circ\varepsilon_\rho$ extends to a positive normalized Markov trace on B_∞ which can be rescaled to be invariant under the Markov moves, and therefore gives rise to a link invariant.

We shall now present the analogue of the Reshetikhin-Turaev construction of topological invariants of compact three-dimensional manifolds [44] and a similar proposal by Wenzl [45]. Our construction is in terms of mixed statistics operators $\varepsilon(\rho_\alpha,\rho_\beta)$ involving all endomorphisms of Δ_{red}, i.e. all superselection sectors of the theory. We have to assume Δ_{red} to be finite ("rational theories"). The construction below formally looks like a trace "$\mathrm{tr}_{\rho_{\mathrm{reg}}}$" w.r.t. a "regular representation $\rho_{\mathrm{reg}}\simeq\oplus_\alpha d_\alpha\rho_\alpha$" (see below). Note, however, that for non-integer dimensions d_α an endomorphisms ρ_{reg} does not exist. A promising object to study might be the state

$\omega_{\text{reg}}(A) = \Sigma_\alpha d_\alpha \omega_\alpha(A)$. In [45] an object with the correct *relative* multiplicities is obtained in a limit of infinite powers of some generating sector.

We expect the superselection sectors and their statistics to be a dual structure w.r.t. an underlying quantum symmetry [24, 25]. In particular, the R- and D-matrices play the group-theoretical role of $6j$-symbols associated with commutativity and associativity of tensor products of representations of this — in general unknown — symmetry. Therefore, we consider the invariants constructed below as characterizations for the full-fledged local quantum field theory and its intrinsic quantum symmetry by invariants labelled by topological objects [46]. The values of the invariants are in principle observable quantities (which might, e.g., be measured in plektonic scattering processes), providing information about the internal symmetry and its correlation with space-time structures. This point of view is completely opposite to the original motivation of [44, 47] which consider their invariants as a scheme to define a variety of "topological field theories" labelled by quantum symmetries, in particular Witten's topological Chern-Simons theories [48].

First recall [3] that the mixed statistics operators give rise to a unitary representation of the groupoid of colored braids, where the colors take values in Δ_{red}. For a coloring $\Lambda = (\rho_1, \ldots, \rho_n)$ of a braid b, $\varepsilon(\Lambda, b)$ is an intertwiner: $\rho_1 \cdots \rho_n \to \rho_{\pi^{-1}(1)} \cdots \rho_{\pi^{-1}(n)}$, where $\pi \in S_n$ is the image of $b \in B_n$ under the natural homomorphism of the braid group onto the permutation group.

Now consider the ribbon braid group $RB_n \supset B_n$ which has besides the braiding generators σ_i of B_n twist generators τ_i $(i = 1 \ldots n)$ with relations $\tau_i \tau_j = \tau_j \tau_i$ and $\tau_i \sigma_i^{\pm 1} = \sigma_i^{\pm 1} \tau_{i+1}$. RB_n is the semidirect product of B_n with $\mathbb{Z}^n$, where $b \in B_n$ acts on $\mathbb{Z}^n$ by the permutation π. The above representation of the groupoid of colored braids extends to a representation of the groupoid of colored ribbon braids by the assignment $\tau_i \mapsto \kappa_\rho$ if ρ is the color of the ribbon to which the twist τ_i applies. Again, the representative $\varepsilon(\Lambda, \beta)$ of a ribbon braid β with coloring Λ is an intertwiner: $\rho_1 \ldots \rho_n \to \rho_{\pi^{-1}(1)} \ldots \rho_{\pi^{-1}(n)}$.

For a given ribbon braid $\beta \in RB_n$ consider the cycle decomposition of the associated permutation $\pi \in S_n$. If the coloring Λ is constant on the cycles (i.e. $\rho_{\pi(i)} = \rho_i$), then $\varepsilon(\Lambda, \beta)$ is a self-intertwiner, and is mapped into a scalar $\text{tr}(\Lambda, \beta)$ by the left-inverse $\phi_n \ldots \phi_1$. This operation is meaningless if Λ is not constant on the cycles. Note that geometrically, if a (ribbon) braid is closed into a (ribbon) link, then every cycle of the permutation corresponds to a component of the link, i.e. Λ constant on the cycles are in fact colorings of the components of the link. The map $\text{tr}(\Lambda, \beta)$ is invariant under groupoid conjugation with any colored ribbon braid, since the same holds for the restriction to colored braids [3] while the phases κ_ρ for the twist generators cancel on both sides.

We observe that the generalization of tr_ρ for *reducible* ρ does not yield a Markov trace on B_∞. However, if tr_ρ is extended to RB_∞ by $\tau_1 \mapsto \Sigma_\alpha \kappa_\alpha E_\alpha$ (the unitary part of the statistics parameter (2.9)), then tr_ρ is a ribbon Markov trace and yields an invariant of framed links. It has the following decomposition in terms of the irreducible subsectors $[\rho_\alpha]$ contributing to ρ with multiplicity N_ρ^α (cf. [46])

$$\mathrm{tr}_\rho(\beta) = d_\rho^{-n} \cdot \sum_\Lambda \left(\prod_c (d_{\alpha(c)})^{l(c)} N_\rho^{\alpha(c)} \right) \cdot \mathrm{tr}(\Lambda, \beta)\,, \tag{B.1}$$

where the sum extends over all colorings of the cycles, the product extends over all cycles, $l(c)$ is the length of the cycle, $\alpha(c)$ is the color assigned to the cycle, and $d_\rho = \Sigma_\alpha N_\rho^\alpha d_\alpha$. We have displayed this formula in order to give the following definition an intuitive interpretation in terms of "ρ_{reg}", although a field-theoretical meaning of such an object is missing.

We define a trace $RB_n \to \mathbb{C}$ by replacing in (B.1) the multiplicities N_ρ^α by d_α:

$$\mathrm{tr}(\beta) := D^{-n} \cdot \sum_\Lambda \left(\prod_c (d_{\alpha(c)})^{l(c)+1} \right) \cdot \mathrm{tr}(\Lambda, \beta)\,, \tag{B.2}$$

where $D := \Sigma_\alpha d_\alpha^2$ is like the "statistical dimension of ρ_{reg}". This functional has remarkable properties.

Proposition B.1. tr *extends to a normalized trace on RB_∞ satisfying*

$$\begin{aligned}
&\text{(a)}\ \ \mathrm{tr}(\beta_1\beta_2) = \mathrm{tr}(\beta_1)\mathrm{tr}(\beta_2) \quad \textit{if } \beta_1, \beta_2 \textit{ are disjoint}\\
&\text{(b)}\ \ \mathrm{tr}(\beta\sigma_n) = D^{-1} \cdot \mathrm{tr}(\beta\tau_n) \quad \textit{if } \beta \in RB_n\\
&\text{(c)}\ \ \mathrm{tr}(\sigma^{(k)}\beta) = D^{-k} \cdot \mathrm{tr}(\tau)\mathrm{tr}(\beta) \quad \textit{if } \beta \in \alpha^{k+1}(RB_\infty)\\
&\text{(d)}\ \ \mathrm{tr}(\beta^{-1}) = \mathrm{tr}(\beta)^*\,.
\end{aligned} \tag{B.3}$$

Here, $\sigma^{(k)} = (\sigma_k \dots \sigma_{2k-1})(\sigma_{k-1} \dots \sigma_{2k-2}) \dots (\sigma_1 \dots \sigma_k)$ is the "k-fold cabled" braiding generator. α is the shift endomorphism $\sigma_i \mapsto \sigma_{i+1}$, $\tau_i \mapsto \tau_{i+1}$. "Disjoint" means that $\beta_1 \in RB_n$ and $\beta_2 \in \alpha^n(RB_\infty)$ or vice versa.

Proof. Except for (c), all the above statements follow from the corresponding properties of $\mathrm{tr}(\Lambda, \beta)$. Note that for disjoint braids the sum over coloring factorizes, and that every unbraided untwisted ribbon (corresponding to a cycle of length 1) contributes to the sum a factor $\Sigma_\alpha d_\alpha^2 = D$.

It remains to prove (c): Let $\beta = \alpha^{k+1}(\beta')$. Since $\alpha|_{RB_n}$ is inner in RB_∞, $\mathrm{tr}(\beta) = \mathrm{tr}(\beta')$. Let $\varepsilon_\Lambda(\beta')$ denote the self-intertwiner: $\rho_1 \dots \rho_n \to \rho_1 \dots \rho_n$ corresponding to the coloring Λ. As compared with β', $\beta\sigma^{(k)}$ has one more cycle of length 2, and for every $i = 1 \dots k-1$ the length of the cycle of β' containing ρ_i is increased by 1 in $\beta\sigma^{(k)}$. Therefore,

$$\mathrm{tr}(\beta\sigma^{(k)}) = D^{-n-k-1} \sum_{\Lambda,\rho} d_\rho^3 \prod_c (d_{\alpha(c)})^{l(c)+1} \prod_{i=1}^{k-1} d_i \cdot \mathrm{tr}((\Lambda, \rho), \beta\sigma^{(k)})\,.$$

We evaluate the trace on the r.h.s. (with the abbreviations $\rho_\otimes = \rho_1 \dots \rho_{k-1}$ and $\phi_\otimes = \phi_{k-1} \dots \phi_1$):

$$\begin{aligned}
&\phi_n \dots \phi_k \phi_\otimes \phi_\rho \phi_\otimes \phi_\rho [\rho\rho_\otimes \rho(\varepsilon_\Lambda(\beta')) \cdot \varepsilon(\rho\rho_\otimes, \rho\rho_\otimes)]\\
&= \phi_n \dots \phi_k \phi_\otimes [\varepsilon_\Lambda(\beta') \cdot \phi_\rho \phi_\otimes \phi_\rho (\rho(\varepsilon(\rho, \rho_\otimes)\varepsilon(\rho, \rho)\rho^2(\varepsilon(\rho_\otimes, \rho_\otimes))\rho(\varepsilon(\rho_\otimes, \rho)))]\,.
\end{aligned}$$

The intertwiner multiplying $\varepsilon_\Lambda(\beta')$ in [...] can be simplified to

$$\begin{aligned}&\lambda_\rho \cdot \phi_\rho \phi_\otimes[\varepsilon(\rho, \rho_\otimes)\rho(\varepsilon(\rho_\otimes, \rho_\otimes))\varepsilon(\rho_\otimes, \rho)] \\ &= \lambda_\rho \cdot \phi_\rho\phi_\otimes[\rho_\otimes(\varepsilon(\rho_\otimes, \rho))\varepsilon(\rho_\otimes, \rho_\otimes)\rho_\otimes(\varepsilon(\rho, \rho_\otimes))] = \lambda_\rho \cdot \phi_\rho[\varepsilon(\rho_\otimes, \rho)\lambda_{\rho_\otimes}\varepsilon(\rho, \rho_\otimes)]\,.\end{aligned}$$

Now, $\lambda_\rho = \kappa_\rho/d_\rho$ and by (2.9) $\lambda_{\rho_\otimes} = (1/d_1 \dots d_{k-1})\Sigma_\xi \kappa_\sigma T_\xi T_\xi^*$, where T_ξ are bases of path intertwiners: $\sigma \to \rho_\otimes$, $\sigma \in \Delta_{\text{red}}$. We proceed with (2.2) and (3.7 + 8)

$$\begin{aligned}&\phi_\rho[\varepsilon(\rho_\otimes, \rho)T_\xi T_\xi^* \varepsilon(\rho, \rho_\otimes)] = \phi_\rho[\rho(T_\xi)\varepsilon(\sigma, \rho)\varepsilon(\rho, \sigma)\rho(T_\xi^*)] \\ &= T_\xi \phi_\rho[\varepsilon(\sigma, \rho)\varepsilon(\rho, \sigma)]T_\xi^* = T_\xi \Sigma_\alpha N_{\sigma\rho}^\alpha(\kappa_\alpha/\kappa_\sigma\kappa_\rho)(d_\alpha/d_\sigma d_\rho)T_\xi^*\,.\end{aligned}$$

Therefore, collecting the terms,

$$\begin{aligned}\text{tr}(\beta\sigma^{(k)}) = D^{-n-k-1}&\sum_\Lambda \left(\prod_c (d_{\alpha(c)})^{l(c)+1}\right) \sum_\xi \left(\sum_{\alpha,\rho} N_{\sigma,\rho}^\alpha \kappa_\alpha \frac{d_\alpha d_\rho}{d_\sigma}\right) \\ &\cdot \phi_n \dots \phi_1(\varepsilon_\Lambda(\beta')T_\xi T_\xi^*)\,.\end{aligned}$$

The essential point is that the sum over α, ρ is independent of the range σ of ξ:

$$d_\sigma^{-1} \sum_\alpha \kappa_\alpha d_\alpha \sum_\rho N_{\bar\sigma\alpha}^\rho d_\rho = d_\sigma^{-1} \sum_\alpha \kappa_\alpha d_\alpha d_\sigma d_\alpha = \sum_\alpha \kappa_\alpha d_\alpha^2\,.$$

Therefore the sum over ξ can be carried out: $\sum_\xi T_\xi T_\xi^* = 1$, and we get

$$\text{tr}(\beta\sigma^{(k)}) = D^{-k-1}\left(\sum_a \kappa_\alpha d_\alpha^2\right) \cdot \text{tr}(\beta)\,.$$

Finally computing $\text{tr}(\tau) = D^{-1}\Sigma_\alpha \kappa_\alpha d_\alpha^2$ establishes (c).

Corollary B.2. *For $\beta \in RB_n$ denote by c the number of cycles of the associated permutation. Assume $tr(\tau) = D^{-1}\Sigma_\alpha\, \kappa_\alpha d_\alpha^2 \neq 0$. Then the functional*

$$\Gamma(\beta) := D^{n-c}\text{tr}(\tau)^{-c} \cdot \text{tr}(\beta) \tag{B.4}$$

is invariant under the moves

$$\begin{aligned}&\text{(C)}\;\; RB_n \ni \beta_1\beta_2 \leftrightarrow \beta_2\beta_1 \in RB_n \\ &\text{(M)}\;\; RB_n \ni \beta\tau_n^{\pm 1} \leftrightarrow \beta\sigma_n^{\pm 1} \in RB_{n+1} \\ &\text{(K)}\;\; RB_n \ni \beta \leftrightarrow \alpha^{k+1}(\beta)\sigma^{(k)\pm 1} \ni RB_{n+k+1} \quad (n \geq k-1)\,.\end{aligned} \tag{B.5}$$

The rescaling (B.4) violates the compatibility with $RB_n \subset RB_{n+1}$. Therefore, Γ is not a trace on RB_∞. The cyclic and Markov invariances (C) and (M) make Γ a (multiplicative) invariant for ribbon links (framed links). Now, every 3-manifold M can be obtained by "surgery along a framed link" embedded in S^3, and two framed links give rise to topologically identical manifolds iff they are related by a sequence of Kirby moves (K) on the closures of ribbon braids [49, 44]. The Kirby moves given in [44] are rather

$$(K') \quad RB_n \ni \beta \leftrightarrow \alpha(\beta \cdot (\sigma_1 \dots \sigma_{k-2})^{k-1}\tau_1 \dots \tau_{k-1}) \\ \cdot \sigma_1 \dots \sigma_{k-1}\sigma_{k-1} \dots \sigma_1\tau_1 \in RB_{n+1}$$

differing from (K) by a sequence of (M) moves. We conclude:

Corollary B.3. *The functional* Γ *is a topological invariant of 3-manifolds.*

A comment is in order for the condition $\mathrm{tr}(\tau) \neq 0$ in B.2. In [20] the formula $|\Sigma_\alpha \kappa_\alpha d_\alpha^2|^2 = \Sigma_\alpha d_\alpha^2$ was proven in the case where there are no "degenerate" sectors besides the vacuum, i.e. sectors which have trivial monodromy with all sectors of the theory. The same calculation in the degenerate case yields

$$\left|\sum_\alpha \kappa_\alpha d_\alpha^2\right|^2 = \left(\sum_\alpha d_\alpha^2\right) \cdot \left(\sum_m \kappa_m d_m^2\right),$$

where the sum Σ_m extends over the sub-category of degenerate sectors. The latter sum vanishes if there are any fermionic degenerate sectors, and equals 1 iff the vacuum sector is the only degenerate one. Therefore, we find that $\mathrm{tr}(\tau) = 0$ iff there are fermionic degenerate sectors (in particular in fermionic theories with permutation group statistics), and the condition $|\mathrm{tr}(\tau)|^2 = D^{-1}$ is equivalent to the absence of degenerate sectors. This purely numerical condition on the statistical dimensions and phases (cf. Cor. 5.2(ii)) in turn, is equivalent [20] to Verlinde's modular algebra [19] satisfied by the matrix S given by the DHR theory in terms of monodromies and the diagonal matrix T of statistics phases.

We note that, in contrast to the original construction [44], we do not require invertibility of the matrix S. But in the degenerate case our invariant gives less information about the 3-manifold. For example, in the bosonic permutation group statistics case, the trace is just $\mathrm{tr}(\beta) = D^{c-n}$, and the invariant Γ is trivial.

We remark that the quantum field theoretical coupling constants $D_{f,e}^{g \circ h}$ of the DHR theory also give rise to an invariant of 3-manifolds of the Turaev-Viro type [47], generalized to the non-self-conjugate case with unrestricted multiplicities $N_{\alpha\beta}^\gamma$. The partition function is a sum over tetrahedra with oriented edges labelled by $\rho_\alpha \in \Delta_{\mathrm{red}}$ and faces labelled by intertwiner bases e according to the labels of the surrounding edges. Every tetrahedron is weighted by a $6j$-symbol ($\propto$ D-matrix element), the edges are weighted by d_α, and the vertices by D^{-1}. The partition function is in fact independent of the edge orientations, since the group S_4 of (admissible) charge conjugations on the edges is implemented by the symmetries of $6j$-symbols (A.5(c)) and further identities of the same type related to $e \mapsto \hat{\bar{e}}$ of type $(\bar{\rho}, \bar{\alpha}, \bar{\beta})$. The Racah-Elliot relations for the D-matrices (see Sec. 3) guarantee invariance under subdivisions of tetrahedra. This invariant W is always non-trivial: $W(S^3) = D^{-1}$, $W(S^2 \times S^1) = 1$. In particular, a relation $W \propto |\Gamma|^2$ may be expected only in the non-degenerate case. We omit the detailed combinatorics.

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