REMARKS ON QUANTUM SYMMETRY*)

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We discuss the concept of Quantum Symmetry in quantum field theory, and in particular the role of the gauge principle. We present a scheme how quantum symmetries can be realized in a Hilbert space, and sketch its construction from the theory of superselection sectors of the gauge invariant (observable) quantities. The approach is independent of (Drinfeld's) quantum groups.

1. Introduction

Four ingredients seem indespensable for a quantum symmetry in quantum field theory. First, the dynamical degrees of freedom are described in terms of a *field algebra* $\mathcal{F}$. Second, there is a *symmetry* $\mathcal{G}$ (e.g., a group, a Lie algebra, or a Hopf algebra). The elements of the symmetry $\mathcal{G}$ act on $\mathcal{F}$. For this action to deserve the name "symmetry" it must preserve the algebraic structure of $\mathcal{F}$ which encodes the underlying physical interpretation of the field operators. On the other hand, acting with different elements of $\mathcal{G}$ will respect (or define) the algebraic structure of $\mathcal{G}$ (as a group, Lie algebra, Hopf algebra, ...).

Third, if we do not want to depart from the probability interpretation of quantum theory, we need a *Hilbert space* $\mathcal{H}$ on which both the field algebra $\mathcal{F}$ and the symmetry $\mathcal{G}$ are represented, as well as the star operation (adjoint) in order to guarantee real expectation values for the self-adjoint operators. These representations *implement* the abstract action $\alpha_g(F)$ of $\mathcal{G}$ on $\mathcal{F}$ in terms of commutation relations among the corresponding representing operators, e.g., for tensor operators ψ_i

$$g\psi_i g^{-1} = \alpha_g(\psi_i) \equiv \sum_j \psi_j \tau_{ji}(g) \tag{1.1}$$

or

$$Ad_x(\psi_i) \equiv \sum_a x_a \psi_i S(x^a) = \alpha_x(\psi_i) \equiv \sum_j \psi_j \tau_{ji}(x) \tag{1.2}$$

in the case of group resp. Hopf algebra symmetries. Here τ_{ji} are finite-dimensional matrix representations of $\mathcal{G}$ and $\sum_a x_a \otimes x^a = \Delta(x)$ and S denote the coproduct and the antipode. These equations may equivalently be written in the form

$$x \cdot \psi_i = \sum_j \psi_j \cdot \sigma_{ji}(x) \tag{1.3}$$

where $\sigma_{ji} := (\tau_{ji} \otimes id) \circ \Delta$ are homomorphisms of $\mathcal{G}$ into the $\mathcal{G}$-valued matrices.

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The fourth, and most important, ingredient is the *gauge principle:* among the elements of $\mathcal{F}$, the action of the symmetry $\mathcal{G}$ distinguishes the gauge invariant elements. This subalgebra $\mathcal{A} = \mathcal{F}^{\mathcal{G}}$ contains the observable part of the theory. The gauge principle brings with it the possibility of superselection sectors: suppose $\Phi \in \mathcal{H}$ to be a gauge-invariant vector, and ψ_i to be some tensor multiplet in $\mathcal{F}$ transforming according to (1.1–3) in some nontrivial representation τ of $\mathcal{G}$. Then all vectors of $\mathcal{H}$ of the form $A\Phi$, where $A \in \mathcal{A}$ is gauge invariant, are also gauge invariant, while all vector multiplets $A\psi_i\Phi$ transform in the representation τ. In particular, the subspaces $\mathcal{H}_0 = \overline{\mathcal{A}\Phi}$ and $\mathcal{H}_\tau = \overline{\oplus_i \mathcal{A}\psi_i\Phi}$ of $\mathcal{H}$ are disjoint, and in general inequivalent, representation spaces of $\mathcal{A}$. In other words: they belong to different superselection sectors of the observables.

One could now address the classification problem: find all *quadruples* $(\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{A} = \mathcal{F}^{\mathcal{G}})$ such that $\mathcal{A}$ is a reasonable theory of physical observables. At this point one should add further physical criteria like Locality, Poincaré covariance, or positivity of the energy spectrum in the Hilbert space $\mathcal{H}$. In particular, $\mathcal{A}$ describes a *local* theory if it is generated by its subalgebras of local operators corresponding to measurements of finite laboratories, which commute with each other at space-like distance. Requiring locality for the gauge invariant operators yields a localization concept for the charged fields: $F \in \mathcal{F}$ is said to be localized somewhere if it commutes with all observables at space-like distance. This does not imply that localized fields are themselves local: Fermi fields anti-commute, but commute with the $\mathbb{Z}_2$ invariant operators of even Fermi number. More interestingly, a $\mathbb{Z}_N$ symmetry admits *anyonic* space-like commutation relations among fields of charges α, β

$$\psi^\alpha(x)\psi^\beta(y) = \psi^\beta(y)\psi^\alpha(x)\, \mathrm{e}^{\pm 2\pi \mathrm{i} Q_\alpha Q_\beta / N} \tag{1.4}$$

and still such fields locally commute with the $\mathbb{Z}_N$ invariant observables. Here $Q_\alpha = \tau^\alpha(Q) = 0, 1 \ldots (N-1)$ are the $\mathbb{Z}_N$ charges of the involved fields, thus expressing the coefficients in terms of symmetry operators. We are most interested in the non-abelian generalizations of anyons: *plektons*.

Instead of aiming at a full classification of possible quantum symmetries, we shall be less ambitious and present a scheme for plektonic field theory [1, 2] which applies to a large class of observable algebras with superselection sectors. To be precise, we pose ourselves the following problem:

Let a specific quantum field theory of local observables, $\mathcal{A}$, be given, and therefore in principle also the knowledge of all its superselection sectors (i.e. equivalence classes of representations with positive energy, possibly distinguished by further physical admissibility criteria). Then we ask for a *triple* $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ which reproduces $\mathcal{A} = \mathcal{F}^{\mathcal{G}}$ and all its sectors by the gauge principle as described above.

This problem was solved very comfortingly [3] in the case of permutation group statistics. In this case, the symmetry can always be established as a compact group of automorphisms of the field algebra, unitarily implemented on the Hilbert space.

2. Quantum symmetry via bimodules

We shall present below a construction of the action of a quantum symmetry in terms of *bimodules* of a specific algebra $\mathcal{G} = \mathcal{M}$. Before we enter into the construction, we want to describe the scheme abstractly.

The algebra $\mathcal{M}$ is the universal hyperfinite type II_1 factor (in the classification of von Neumann algebras: i.e., there is a trace state tr on $\mathcal{M}$ which on the projections of $\mathcal{M}$ can take all values between 0 and 1). It is only the class of the *relevant* bimodules of this algebra which distinguishes different theories. Note that every endomorphism $\rho : \mathcal{M} \to \mathcal{M}$ induces a bimodule action of $\mathcal{M}$ on itself by left and right multiplication with $\rho(m)$. This bimodule, coinciding as a space with $\mathcal{M}$ will be denoted by $\mathcal{M}_\rho$. The central point in our construction is that the analysis of the superselection sectors of a given local theory distinguishes the class of endomorphisms of $\mathcal{M}$ relevant for the quantum symmetry in 1:1 correspondence with the sectors.

The bimodules $\mathcal{M}_\rho$ are in general not irreducible. In particular, every pair of projections in $\mathcal{M}$ commuting with $\rho(\mathcal{M})$ defines an invariant sub-bimodule $E_\alpha \mathcal{M}_\rho E_\beta$. It turns out convenient to pick $\rho_\oplus$ to contain all equivalence classes of relevant irreducible endomorphisms ρ_α precisely once (i.e., there corresponds a unique minimal projection E_α in the commutant of $\rho_\oplus$), and to distinguish the projection E_0 referring to the trivial endomorphism $\rho_0 = id$. Then $\mathcal{M}_\alpha = E_\alpha \mathcal{M}_{\rho\oplus} E_0$ exhaust the *irreducible* bimodules. It is further convenient to use a natural identification of $\mathcal{M}_\alpha$ with linear spaces $\mathcal{K}_\alpha$ under which the two-sided action on $\mathcal{M}_\alpha$ given by $E_\alpha \rho_\oplus(m) x \rho_\oplus(n) E_0$ turns into $\rho_\alpha(m)kn$. More generally, for all relevant endomorphisms one can introduce $\mathcal{K}_\rho$ such that

$$\mathcal{K}_\rho \ni k \mapsto \rho(m) \cdot k \cdot n \qquad (m, n \in \mathcal{M}). \tag{2.1}$$

Now let $\{b_I\}$ denote a finite left basis of $\mathcal{K}_\rho$ such that every element $k \in \mathcal{K}_\rho$ has an expansion

$$k = \sum_I b_I \cdot (b_I, k) \tag{2.2}$$

with an $\mathcal{M}$-valued inner product (k, l). Then the action (2.1) is rewritten

$$\rho(m) \cdot b_I = \sum_j b_j \cdot \rho_{JI}(m) \tag{2.3}$$

where $\rho_{JI}(m) = (b_J, \rho_\alpha(m) b_I)$ are a matrix of coefficients in $\mathcal{M}$. In fact, $\rho_{\text{mat}} = (\rho_{JI})$ is a homomorphism from $\mathcal{M}$ into the $\mathcal{M}$-valued matrices, and the number $d(\rho) = \sum_I \text{tr}(\rho_{II}(1))$ coincides with the von Neumann dimension of $\mathcal{K}_\rho$ as a left module.

Bimodules can be tensored with each other by identifying the left action on the right factor with the right action on the left factor: $(k_1 \cdot m) \otimes k_2 \approx k_1 \otimes \rho_2(m) \cdot k_2$. We denote the bimodule tensor product by $\otimes_\mathcal{M}$ and find that it can be written in the form

$$k_1 \otimes \mathcal{M} k_2 = \rho_2(k_1) k_2 \tag{2.4}$$

where multiplication and application of ρ on elements of $\mathcal{K}_\rho$ are inherited from the above mentioned identifications of $\mathcal{K}_\rho$ with subspaces of $\mathcal{M}$, where these operations do have a meaning. In particular, by tensoring $\rho_1(m)k_1$ with k_2 we find that

$$\mathcal{K}_{\rho_1} \otimes_{\mathcal{M}} \mathcal{K}_{\rho_2} = \mathcal{K}_{\rho_2\rho_1} \tag{2.5}$$

where $\rho_2\rho_1$ is the composition of endomorphisms.

Let $\varepsilon(\rho,\sigma)$ be unitary operators which intertwine the product endomorphisms $\rho\sigma$ and $\sigma\rho$. These operators naturally also intertwine the corresponding tensor product bimodules $\mathcal{K}_\sigma \otimes_{\mathcal{M}} \mathcal{K}_\rho$ and $\mathcal{K}_\rho \otimes_{\mathcal{M}} \mathcal{K}_\sigma$. (Note the different order of the factors!) Indeed, such intertwiners ("statistics operators") are provided by the theory of superselection sectors as the intrinsic characterization of generalized particle statistics [4], and are naturally lifted to intertwine also the endomorphisms of $\mathcal{M}$. Therefore, the tensor product of the bimodules under consideration is independent of the order of factors up to unitary equivalence. The (lifted) statistics operators replace the "universal R-matrix" of quasi-triangular Hopf algebras.

Similarly, intertwiners in the theory of superselection sectors which single out irreducible subsectors from a product of superselection sectors, lift to intertwiners between endomorphisms $\rho_\alpha\rho_\beta$ and ρ_γ of $\mathcal{M}$, and yield a decomposition of the corresponding bimodules

$$\mathcal{K}_\beta \otimes_{\mathcal{M}} \mathcal{K}_a \simeq \bigoplus_\gamma N^\gamma_{\alpha\beta}\mathcal{K}_\gamma \tag{2.6}$$

with the same multiplicities $N^\gamma_{\alpha\beta}$ as the "fusion rules" of the superselection structure. They play the role of Clebsch-Gordan decomposition of tensor products of matrix representations of Hopf algebras.

So far for the *abstract* symmetry structure. The next issue in the construction below is an isometric and unital embedding $k \mapsto \psi_k$ of $\mathcal{K}_\rho$ into the field algebra, i.e.,

$$\psi_k^*\psi_l = (k,l) \qquad \text{and} \qquad \sum_I \psi_I\psi_I^* = 1. \tag{2.7}$$

The latter equation holds for any basis (2.2), with $\psi_I \equiv \psi_{b_I}$. Since (k,l) is in $\mathcal{M}$, this means in particular that also $\mathcal{M}$ is contained in the field algebra (actually, $\mathcal{M} \subset \mathcal{F}$ is the isometric image of $\mathcal{K}_0$). The bimodule symmetry is implemented in the field algebra by

$$m \cdot \psi_k \cdot n = \psi_{\rho(m)kn} \tag{2.8}$$

and therefore, in terms of a basis $\psi_I = \psi_{b_I}$

$$m \cdot \psi_I = \sum_J \psi_J \cdot \rho_{JI}(m). \tag{2.9}$$

This formula naturally generalizes the transformation law (1.3). (ρ_{JI}) can in general not be written in terms of a coproduct and a numerical matrix representation.

By working out the details of our construction, we find that by virtue of the isometric embeddings of bimodules $\mathcal{K}_\rho$ into the field algebra, the previously mentioned structures of exchange and reduction of bimodule tensor products turn into

$$\psi_I^\alpha(x)\psi_J^\beta(y) = \sum_{J'I'} \psi_{J'}^\beta(y)\psi_{I'}^\alpha(x) \cdot \mathcal{R}_{JI}^{I'J'}(\pm) \tag{2.10}$$

$$\psi_I^\alpha \psi_J^\beta = \sum_{e,K} T_e \cdot \psi_K^\gamma \cdot \mathcal{C}_e \left(\begin{array}{c|c} \gamma & \beta\,\alpha \\ K & J\,I \end{array} \right) \tag{2.11}$$

where both the $\mathcal{R}$-matrix and the Clebsch-Gordan coefficients $\mathcal{C}$ take values in $\mathcal{M}$, $\mathcal{R}$ satisfies the appropriate braid group identity involving ρ_{mat} as shift operation. x and y refer to translation such that $\psi^\alpha(x)$ and $\psi^\beta(y)$ become localized at space-like distance, the sign $\pm$ depending on x being to the left or to the right of y. The sum over e in (2.11) extends over the multiplicity of $\mathcal{K}_\gamma$ being contained in $\mathcal{K}_\alpha \otimes_{\mathcal{M}} \mathcal{K}_\beta$. T_e are gauge-invariant operators, i.e. local observables.

We conclude that our scheme of quantum symmetry via bimodules is realized in terms of commutation relations (2.9) within the field algebra, and provides specific exchange and Clebsch-Gordan expansion formulae. There is finally a conjugation structure $\mathcal{K}_\rho \ni k \mapsto k^+ \in \mathcal{K}_{\bar\rho}$ (inherited from * in $\mathcal{M}$) such that

$$(\rho(m)kn)^+ = \bar\rho(n^*)k^+m^* \tag{2.12}$$

which is implemented by the Hilbert space adjoint in the field algebra

$$\psi_k^* = (\mathit{gauge\ invariant}) \cdot \psi_{k^+}. \tag{2.13}$$

It plays the role of the conjugate representation of Hopf algebras induced by the antipode. Therefore, all of the structural ingredients of quasi-triangular Hopf algebras (quantum groups) have their counterpart in the bimodule approach, and have the correct physical meaning in the algebra of charged fields.

3. Strategy of the construction

The construction of the triple $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ from the local quantum field theory $\mathcal{A}$ [1, 2] proceeds in four major steps, which we shall sketch only cursorily. The first step is the analysis of the superselection structure of $\mathcal{A}$. There exists a fully elaborated theory of superselection sectors [4]. Asymptotically vacuum-like positive energy representations of $\mathcal{A}$ are described in terms of localized endomorphisms ρ of $\mathcal{A}$. The composition of representations is established by the product of endomorphisms. This product is non-commutative, but $\rho\sigma$ and $\sigma\rho$ differ only up to some inner unitary similarity transformation given by the statistics operator $\epsilon(\rho, \sigma)$. Endomorphisms which possess a conjugate have finite local index $\mathrm{Ind}[\rho(\mathcal{A}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O})] = d(\rho)^2$ [5], defining a dimension function $d(\rho)$. They are the objects of a strict monoidal braided C^* category with direct sums, subobjects, and conjugates (for short: BC^*C) [2]. The (monoidal) product of objects is the composition of endomorphisms, and the C^*, braided, and conjugation structures are provided by the theory of superselection sectors. In particular, every product of irreducible representations is finitely reducible and, with $N^{\gamma}_{\alpha\beta}$ the fusion multiplicities of the theory,

$$d(\rho_\alpha)d(\rho_\beta) = \sum_\gamma N^{\gamma}_{\alpha\beta} d(\rho_\gamma). \tag{3.1}$$

In the second step, one constructs from this BC^*C a (model of the universal) hyperfinite type II_1 von Neumann factor $\mathcal{R}$, denoted by $\mathcal{M}$ in the present context. At this point one needs for technical reasons the assumption that the number of sectors in the theory is finite ("rational theories"), but we expect that this condition may be relaxed. $\mathcal{M}$ is the "path model" [6, 7] of pairs of paths (strings) (ξ, η) with common initial and end points, equipped with the Witten "string product". The paths live in a graph with vertices α and $N^{\gamma}_{\alpha\beta}$ edges of color α extending between the vertices β and γ. The trace on $\mathcal{M}$ is provided by a conditional expectation associated with the conjugation of sectors. The essential point now is that the endomorphisms of $\mathcal{A}$ induce endomorphisms of $\mathcal{M}$ (denoted by the same symbol) in terms of a parallel transport in the path model terminology. The corresponding cell matrices [7] are given by certain coupling constants of the superselection theory (fusion matrices) which play a similar role like Wigner-Racah $6j$-symbols for the reduction of multiple tensor products of group representations.

The third step consists in the observation that the bimodules $\mathcal{K}_\rho$ of $\mathcal{M}$ induced by the endomorphisms ρ of $\mathcal{M}$ (see above) themselves are the objects of another BC^*C. The (monoidal) product of this category is the bimodule tensor product (2.5), while the C^*, braided, and conjugation structures are again given by the local interwiners which are naturally lifted to intertwiners between bimodules. The fact remarked above that the bimodule tensor product inverts the order of the factors in the corresponding product of endomorphisms amounts to saying that the two BC^*C's are pseudo-equivalent to each other [2].

The final step is the "hard part": the actual construction of the field algebra into which $\mathcal{K}_\rho$ are isometrically embedded, and of the Hilbert space on which the bimodule action is implemented by (2.9) singling out $\mathcal{A} = \mathcal{F}^{\mathcal{G}} \equiv \mathcal{F} \cap \mathcal{M}'$, the commutant of $\mathcal{M}$ in $\mathcal{F}$. For its detailed description we refer to [1]. It relies heavily on an intrinsic CPT symmetry of the coupling constants of the superselection structure, and involves a sort of "contraction" of the dynamical degrees of freedom given by the endomorphisms of $\mathcal{A}$ with their CPT-reflected inner degrees of freedom given by the bimodules of $\mathcal{M}$. As a result, the field multiplets $\psi_k, k \ni \mathcal{K}_\rho$, are charged fields in the sense that they create the charged sectors from the vacuum sector by virtue of the commutation relations with the observables

$$\psi_k \cdot A = \rho(A) \cdot \psi_k. \tag{3.2}$$

Together with the embedding $k \mapsto \psi_k$ being isometric and unital (2.4), this is a precise generalization of the construction [3] in the case of permutation group statistics, where the symmetry is established as a compact group of automorphisms linearly transforming the finite multiplets ψ_i. The latter are isometric and unital embeddins of vector spaces of complex dimension $d(\rho) \in \mathbb{N}$ (the Cuntz algebra) and extend the algebra of observables by the same relations (3.2).

The Hilbert space obtains by the GNS construction from a ground state on the field algebra, which in turn is induced from the vacuum state on the observables $\mathcal{A}$ and the trace state on $\mathcal{M}$. In restriction to $\mathcal{A} = \mathcal{F}^{\mathcal{G}}, \mathcal{H}$ decomposes into the superselection sectors $\mathcal{H}_\alpha$ with multiplicity spaces V_α (the completions of $\mathcal{K}_\alpha$) on which the symmetry acts:

$$\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha \otimes V_\alpha. \tag{3.3}$$

4. Conclusions

We have seen that alternative schemes than quantum groups are viable in order to describe quantum symmetries in quantum field theory and to explain the occurrence of superselection sectors. Our approach has the advantage of being readily implemented on a Hilbert space without any (intermediate) indefinite norm problems.

Another issue is the following. The bimodule approach involves representation spaces of infinite complex dimension, but it identifies the von Neumann dimensions of these spaces with the intrinsic "quantum" dimensions of the sectors. In contrast, to some given model such as (the chiral version of) the Ising model with its simple fusion rules one can associate several candidate deformations of classical Lie algebras which at the appropriate value of the deformation parameter are "truncated" to yield the same desired finite fusion rules. Which of these quantum groups should then be regarded as the "true" quantum symmetry, and can there be any intrinsic meaning of the *naive* dimensions of its representations? To some extent, the answer has been given in [8]: the truncated quantum group ceases to be a Hopf algebra, and in particular has few if any remembrance of the original classical Lie algebra or its deformation. In fact, as an algebra with only finitely many representations it rather seems to be a generalization of a *finite* group.

One may feel uneasy with our construction, the coefficients of commutation relations and Clebsch-Gordan expansions taking values in the symmetry algebra itself (and hence that the symmetry algebra is a *subalgebra* of the field algebra). But 1° this is not in conflict with any physical principle, and 2° we note that, e.g., symmetry group valued Clebsch-Gordan coefficients are necessary to reconcile commutation relations of the form (1.4) with N replaced by $2N$—encountered in models with simple conformal currents (N even)—with the $\mathbb{Z}_N$ gauge principle suggested by the fusion rules. Here, an obstruction (which occurs only with *braid* group statistics) intrinsic to the algebra of local observables prevents any construction of charged fields with numerical Glebsch-Gordan coefficients [2]. A similar formal structure was obtained by Mack and Schomerus [8] when passing to the quasi Hopf symmetry in order to accomodate the fusion rules of the minimal models. They traced back the non-numerical coefficients to the lack of co-commutativity of the coproduct.

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