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Markov Traces as Characters for Local Algebras

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ABSTRACT: Statistics governs the superselection structure of local quantum field theory. The intrinsic information is encoded in Markov traces, which characterize equivalence classes of irreducible representations (sectors) as well as their composition rules. The Markov traces also provide a tool to classify (quantum) symmetries.

1. Introduction

The analysis of *statistics* is a powerful tool to explore the structure of the physically relevant representation theory of local algebras [1]. It provides information about the *superselection rules*, which reflect the particle or charge content as well as interactions, in intimate relation to the concepts of internal symmetry [2] and covariance (spin).

In four-dimensional massive quantum field theory, permutation group statistics yields very natural and familiar results: the presence of a compact (global) gauge group governing the selection rules, and the usual spin-statistics theorem. In lower dimensions, the occurrence of braid group statistics reflects the possibility of fractional spin and a more general "quantum" notion of symmetry. We want to contribute to a structural understanding of these issues, which goes beyond the empirical study in large classes of conformal models. Although we shall recover most of the structure familiar from these models, we shall never assume or exploit conformal covariance throughout this article.

The starting point is the observation [1], that precisely those *representations* π of a local algebra $\mathcal{A}$, which describe "particle-like" excitations (as compared to the vacuum representation π_0; for the precise setting, we refer to the original literature [1,3]) can be alternatively described in terms of appropriate *endomorphisms* ρ of $\mathcal{A}$, such that π is unitarily equivalent to

$$\pi_\rho = \pi_0 \circ \rho. \tag{1.1}$$

In particular, there is a natural composition law for representations $\pi_i = \pi_0 \circ \rho_i$:

$$\pi_1 \times \pi_2 = \pi_0 \circ \rho_1 \circ \rho_2, \tag{1.2}$$

which comprises the physical connotation of building multi-particle asymptotic states from one-particle states.

The theory of superselection sectors describes the reducibility of product representations $\pi_1 \times \pi_2$, resp. composite endomorphisms $\rho_1\rho_2 \equiv \rho_1 \circ \rho_2$ (the superselection rules). The theory of statistics describes the non-commutativity of the composition law. Both turn out to be closely related to each other, as well as to the notion of charge conjugation.

Since the set of endomorphisms of an algebra is determined by the algebra itself, the study of its composition law is an intrinsic characterization of the algebra. In the quantum field theoretical case at hand, the data of this characterization can be given as trace states on the braid group (or some larger algebra) with special (Markov) properties, i.e. as a collection of class-invariant numbers. It does not *refer* to non-local and unobservable fields creating the representations of interest from the vacuum (hence lying outside the algebra $\mathcal{A}$), but determines the algebraic properties (commutation relations) of would-be such fields, and in fact allows to *construct* them.

The very restrictive properties of the field-theoretical Markov traces offer a handle to *classify* statistics and superselection structures, and thus gives a hint on the laws governing fractional spins and on the nature of quantum symmetry. We shall point out an interpretation of particle-like representations with braid group statistics as *soliton-like* representations of some extended algebra.

2. Statistics

Let us fix some terminology.

The algebra $\mathcal{A}$ is generated by subalgebras $\mathcal{A}(\mathcal{O})$ of operators localized in the simply connected bounded space-time regions $\mathcal{O}$ (which are intervals on the real axis, if "space-time" is the "chiral light-cone" of conformal field theory). $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute if the two regions have space-like separation: $\mathcal{O}_1 \times \mathcal{O}_2$. We call $\mathcal{O}'$ the causal complement of $\mathcal{O}$, which is simply connected if $D > 2$, and which consists of the right causal complement $\mathcal{O}^+$ and the left causal complement $\mathcal{O}^-$ if $D \leq 2$. In the latter case we write $\mathcal{O}_1 < \mathcal{O}_2$ if $\mathcal{O}_1 \subset (\mathcal{O}_2)^-$.

The relevant endomorphisms of $\mathcal{A}$ corresponding to translationally covariant representations are *localized* and *transportable*: ρ is localized in $\mathcal{O}$, iff it acts trivially on the subalgebra $\mathcal{A}(\mathcal{O}')$. ρ is transportable, iff for every translate region $\tilde{\mathcal{O}}$ of $\mathcal{O}$ there is an equivalent endomorphism $\tilde{\rho}$ localized in $\tilde{\mathcal{O}}$. The set of localized and transportable endomorphisms of $\mathcal{A}$ is called Δ_t. (While these properties are physically characteristic of particle-like excitations, mathematically they are necessary conditions for the representation $\pi_\rho = \pi_0 \circ \rho$ to be created by *localized* fields ψ, i.e. $\pi(A)\psi = \psi\pi_0(A)$, see sect.5. There we shall also consider *solitonic* endomorphisms, for which π_ρ in restriction to the left and right causal complements coincide with the same vacuum representation π_0 only upto unitary equivalence.)

Two localized endomorphisms ρ, $\tilde{\rho}$ are equivalent, i.e. $\pi_\rho \simeq \pi_{\tilde{\rho}}$, iff there is a unitary $U \in \mathcal{A}$ such that

$$\tilde{\rho}(A) = U\rho(A)U^*, \quad \text{i.e. } \tilde{\rho} = \sigma_U \circ \rho.$$

Here $\sigma_U \in \Delta_t$ is the inner automorphism

$$\sigma_U(A) = UAU^*$$

localized in $\mathcal{O}$ iff $U \in \mathcal{A}(\mathcal{O})$. Inner automorphisms do not lead to new superselection sectors (equivalence classes of representations).

Whenever $\pi_1 = \pi_0 \circ \rho_1$ and $\pi_2 = \pi_0 \circ \rho_2$ possess common subrepresentations, then there are operators *intertwining* from ρ_2 to ρ_1, i.e. operators $T \in \mathcal{A}$ satisfying

$$T\rho_2(A) = \rho_1(A)T.$$

The linear space of intertwiners from ρ_2 to ρ_1 is denoted by $(\rho_1|\rho_2)$. In particular, $U \in (\sigma_U|id)$.

The composition (1.2) of irreducible representations will in general be *reducible*. Then there are projections E in the commutant $\rho_1\rho_2(\mathcal{A})'$ corresponding to irreducible components $\rho_3 \in \Delta_t$ of $\rho_1\rho_2 \in \Delta_t$, and intertwiners

$$T \in (\rho_1\rho_2|\rho_3), \quad TT^* = E.$$

The dimension of the space of intertwiners $(\rho_1\rho_2|\rho_3)$, $\rho_i \in \Delta_t$ irreducible, is the multiplicity of $\pi_3 = \pi_0 \circ \rho_3$ in $\pi_1 \times \pi_2$, and will be seen to be a class-invariant, i.e. independent of the representatives ρ_i from the equivalence classes (superselection sectors) $[\rho_i]$.

The composition law (1.2) is in general *non-commutative*. But if $\rho_1, \rho_2 \in \Delta_t$ are localized in $\mathcal{O}_1 \times \mathcal{O}_2$, then

$$\rho_1\rho_2 = \rho_2\rho_1, \quad \Leftrightarrow \quad \pi_1 \times \pi_2 = \pi_2 \times \pi_1. \tag{2.1}$$

Let us state the main results [1,3] defining statistics.

2.1. Theorem: There is a collection of unitary statistics operators for every pair $\rho_1, \rho_2 \in \Delta_t$, implementing the unitary equivalence between $\pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$:

$$\pi_2 \times \pi_1(A) = \pi_0(\varepsilon(\rho_1, \rho_2)) \cdot \pi_1 \times \pi_2(A) \cdot \pi_0(\varepsilon(\rho_1, \rho_2))^*,$$

$$\Leftrightarrow \quad \varepsilon(\rho_1, \rho_2) \in (\rho_2\rho_1|\rho_1\rho_2), \tag{2.2}$$

which is uniquely determined by the properties

$$\left.\begin{array}{l} \rho_3(T)\varepsilon(\rho_1, \rho_3) = \varepsilon(\rho_2, \rho_3)T \\ \rho_3(T)\varepsilon(\rho_3, \rho_1)^* = \varepsilon(\rho_3, \rho_2)^*T \end{array}\right\} \text{for } T \in (\rho_2|\rho_1), \tag{2.3}$$

$$\varepsilon(\rho_1, \rho_2) = 1, \quad \text{if} \left\{ \begin{array}{ll} \mathcal{O}_1 \times \mathcal{O}_2 & \text{for } D > 2 \\ \mathcal{O}_2 < \mathcal{O}_1 & \text{for } D \leq 2 \end{array} \right. \tag{2.4}$$

(ρ_i localized in $\mathcal{O}_i$). These properties imply

$$\begin{array}{l} \varepsilon(\rho_1\rho_2, \rho_3) = \varepsilon(\rho_1, \rho_3)\rho_1(\varepsilon(\rho_2, \rho_3)), \\ \varepsilon(\rho_3, \rho_1\rho_2) = \rho_1(\varepsilon(\rho_3, \rho_2))\varepsilon(\rho_3, \rho_1), \end{array} \tag{2.5}$$

$$\begin{array}{l} \rho_3(\varepsilon(\rho_1, \rho_2))\varepsilon(\rho_1, \rho_3)\rho_1(\varepsilon(\rho_2, \rho_3)) = \\ \quad = \varepsilon(\rho_2, \rho_3)\rho_2(\varepsilon(\rho_1, \rho_3))\varepsilon(\rho_1, \rho_2), \end{array} \tag{2.6}$$

and

$$\varepsilon(\rho_1,\rho_2)\varepsilon(\rho_2,\rho_1) = 1, \text{ if } D > 2. \qquad (2.7)$$

(By virtue of the transportability, ρ_i can be written as $\sigma_{U_i^*} \circ \tilde{\rho}_i$, where $\tilde{\rho}_i$ are localized as in (2.4), thus $\varepsilon(\tilde{\rho}_1,\tilde{\rho}_2) = 1$. Then using (2.3), $\varepsilon(\rho_1,\rho_2)$ can be computed in terms of the charge transporting operators U_i. The result is independent of the auxiliaries U_i and $\tilde{\rho}_i$.)

For $D \leq 2$, in general $\varepsilon(\rho_1,\rho_2) \neq 1$ for $\mathcal{O}_1 < \mathcal{O}_2$.

The rules (2.3), (2.5), (2.6) express the commutativity of the following diagrams (with the arrows implemented by the respective intertwiners):

$$\begin{array}{ccc} \pi_1\times\pi_3 & \rightarrow & \pi_2\times\pi_3 \\ \downarrow & & \downarrow \\ \pi_3\times\pi_1 & \rightarrow & \pi_3\times\pi_2, \end{array}$$

$$\begin{array}{ccc} (\pi_1\times\pi_2)\times\pi_3 & \equiv & \pi_1\times\pi_2\times\pi_3 \\ \downarrow & & \downarrow \\ \pi_3\times(\pi_1\times\pi_2) & \leftarrow & \pi_1\times\pi_3\times\pi_2, \end{array}$$

$$\begin{array}{ccccc} \pi_1\times\pi_2\times\pi_3 & \rightarrow & \pi_2\times\pi_1\times\pi_3 & \rightarrow & \pi_2\times\pi_3\times\pi_1 \\ \downarrow & & & & \downarrow \\ \pi_1\times\pi_3\times\pi_2 & \rightarrow & \pi_3\times\pi_1\times\pi_2 & \rightarrow & \pi_3\times\pi_2\times\pi_1. \end{array}$$

2.2. Corollary:

$$\varepsilon_\rho := \varepsilon(\rho,\rho) \in (\rho^2|\rho^2) \equiv \rho^2(\mathcal{A})'. \qquad (2.8)$$

$$\rho(\varepsilon_\rho)\varepsilon_\rho\rho(\varepsilon_\rho) = \varepsilon_\rho\rho(\varepsilon_\rho)\varepsilon_\rho. \qquad (2.9)$$

$$\varepsilon_\rho^2 = 1, \text{ if } D > 2. \qquad (2.10)$$

(Observe that every spectral projection of the statistics operator ε_ρ is a projection in the commutant $\rho^2(\mathcal{A})'$ of ρ^2, hence defines a subrepresentation of $\pi_\rho \times \pi_\rho$. This bearing of the statistics on the reducibility of composite representations will be worked out in detail in sect.3.)

2.3. Corollary: Let B_n denote the braid group generated by the symbols σ_i, $i = 1,\ldots,n-1$, with the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

Let $\rho \in \Delta_t$. Then the mapping

$$\varepsilon_\rho^{(n)} : \sigma_i \mapsto \rho^{i-1}(\varepsilon_\rho) \qquad (2.11)$$

defines a homomorphism of the braid group B_n into the unitary operators in the commutant $(\rho^n|\rho^n) = \rho^n(\mathcal{A})' \subset \mathcal{A}$. The family $\{\varepsilon_\rho^{(n)} | \, n \in \mathbb{N}\}$ is compatible with the inclusions $B_{n-1} \subset B_n$ and $(\rho^{n-1}|\rho^{n-1}) \subset (\rho^n|\rho^n)$, and thus extends to a homomorphism $\varepsilon_\rho^{(\infty)}$ of the infinite braid group B_∞ into $\bigcup_n(\rho^n|\rho^n) \subset \mathcal{A}$.

If $D > 2$, then $\varepsilon_\rho^{(n)}$ respect also the additional relation $\sigma_i^2 = 1$, hence define homomorphisms of the permutation groups S_n into the unitaries in $(\rho^n|\rho^n)$.

(The above statements naturally generalize to the groupoid of "colored braids", involving the statistics operators $\varepsilon(\rho_j,\rho_k)$ instead of $\varepsilon_\rho = \varepsilon(\rho,\rho)$. We shall turn to this extension in the discussion of the identities among field theoretical Markov traces, sect.4.)

3. Superselection Structure

We shall now sketch the relevance af the statistics for the superselection structure, i.e. the reducibility of composite endomorphisms. The results are by now standard, see e.g. the recent reviews [4], and perfectly match with the experience from conformal models. To obtain the results below in the general theory of superselection sectors, it is, however, necessary to exclude sectors the square of which could be infinitely reducible (such sectors cannot occur in massive theories, but might be relevant in "non-rational" conformally covariant theories) and keep only the proper ones. These are sectors for which there is a charge conjugation:

Let $\rho \in \Delta_t$ be irreducible. $\bar{\rho} \in \Delta_t$ irreducible is a *conjugate* of ρ if $\pi_{\bar{\rho}} \times \pi_\rho$ contains the vacuum as an irreducible subrepresentation, i.e. there is an isometry

$$R \in (\bar{\rho}\rho|id), \quad R^*R = 1, \qquad (3.1)$$

and if

$$\lambda(\rho) := R^*\bar{\rho}(\varepsilon_\rho)R \neq 0. \qquad (3.2)$$

The latter quantity is a scalar, called the statistics parameter, its phase ω the statistics phase, and its inverse modulus $d \geq 1$ the statistical dimension:

$$\lambda(\rho) = \frac{\omega(\rho)}{d(\rho)}. \qquad (3.3)$$

(The non-vanishing of the statistics parameter is crucial as a part of the definition of the conjugate, since most of the following properties of conjugates would not hold for

$\lambda = 0$. In fact, $\lambda = 0$ can be excluded for massive covariant representations. Recall that the statistics operator can be computed in terms of translation operators. This also lies at the origin of the relation between spin and statistics:

$$\omega(\rho) = \exp 2\pi i\ s, \tag{3.4}$$

where s is a spin occurring in the representation $\pi_0 \circ \rho$.)

If $\lambda \neq 0$, then ρ and π_ρ are called *proper.*

3.1. Proposition: If ρ is proper, then its conjugate $\bar{\rho}$ is unique up to equivalence. The multiplicity of the vacuum in $\pi_{\bar{\rho}} \times \pi_\rho$ is one, i.e. $dim(\bar{\rho}\rho|id) = 1$. $\lambda(\rho)$ is independent of R, and is in fact a class-invariant of the sector $[\rho]$. ρ is a conjugate of $\bar{\rho}$, $\bar{\rho}$ is proper, and

$$\lambda(\bar{\rho}) = \lambda(\rho). \tag{3.5}$$

Let us now introduce the *left-inverse* ϕ of ρ as the mapping of $\mathcal{A}$ into $\mathcal{A}$

$$\phi(A) := R^* \bar{\rho}(A) R. \tag{3.6}$$

While in general, ϕ is no endomorphism of $\mathcal{A}$, it satisfies

$$\begin{aligned} \phi(1) &= 1, \\ \phi(\rho(A) B \rho(C)) &= A\phi(B)C. \end{aligned} \tag{3.7}$$

In particular, $\phi \circ \rho = id$ (whence its name). If ρ is proper, then ϕ is the unique positive mapping satisfying (3.7).

The left-inverse is very useful to study the superselection structure. Namely, every subrepresentation of $\pi_1 \times \pi_2$ corresponds to a projection E in the commutant $\rho_1\rho_2(\mathcal{A})' = (\rho_1\rho_2|\rho_1\rho_2)$. Exploiting the property of left-inverses:

$$T \in (\rho\rho_\alpha|\rho\rho_\beta) \Rightarrow \phi(T) \in (\rho_\alpha|\rho_\beta),$$

one finds $\phi_1(E) \in \rho_2(\mathcal{A})'$. If ρ_2 is irreducible, this can only be a scalar by Schur's Lemma. These numbers controlling the reducibility of composite endomorphisms can actually be computed:

3.2. Theorem: Let π_1, π_2 be proper. Then $\pi_1 \times \pi_2$ is equivalent to a finite sum of proper subrepresentations $\pi^{(j)} = \pi_0 \circ \rho^{(j)}$. The corresponding minimal orthogonal projections $E^{(j)} \in \rho_1\rho_2(\mathcal{A})'$ have "relative dimensions"

$$\phi_1(E^{(j)}) = \frac{d(\rho^{(j)})}{d_1 d_2}. \tag{3.8}$$

To every component $\rho^{(j)}$ corresponds an eigenvalue of the monodromy operator:

$$\varepsilon(\rho_2, \rho_1)\varepsilon(\rho_1, \rho_2)E^{(j)} = \frac{\omega(\rho^{(j)})}{\omega_1\omega_2} E^{(j)}. \tag{3.9}$$

The sum rule for statistical dimension

$$\sum_j d(\rho^{(j)}) = d_1 d_2 \tag{3.10}$$

holds, and every $d(\rho^{(j)})$ lies in the interval

$$\max\left(\frac{d_1}{d_2}, \frac{d_2}{d_1}\right) \leq d \leq d_1 d_2. \tag{3.11}$$

Endomorphisms need not to be invertible, i.e. automorphisms. If $\rho \in \Delta_t$ is invertible, then its class $[\rho]$ is called a *simple* sector. We give criteria for this situation.

3.3. Proposition: Let $\rho \in \Delta_t$ be proper. The following statements are equivalent.
(i) ρ is an automorphism, i.e. ρ is invertible.
(ii) ρ^2 is irreducible.
(iii) ε_ρ is a scalar.
(iv) $d(\rho) = 1$.

Let ∇ denote the set of equivalence classes $[\rho]$ of proper endomorphisms. The equivalence classes of the proper components of $\rho_\alpha\rho_\beta$ are independent of the representatives $\rho_\alpha \in [\rho_\alpha]$, $\rho_\beta \in [\rho_\beta]$, and coincide with those of $\rho_\beta\rho_\alpha$. Thus it is justified to introduce the class-invariant multiplicities

$$N_{\alpha\beta}^{\gamma} = dim(\rho_\alpha\rho_\beta|\rho_\gamma) \tag{3.12}$$

of $[\rho_\gamma]$ in $[\rho_\alpha][\rho_\beta]$, which may be organized into "fusion matrices" $(N_\alpha)_\beta^\gamma \equiv N_{\alpha\beta}^\gamma$.

3.4. Proposition: For $\alpha \in \nabla$ denote by $\bar{\alpha}$ the conjugate class $[\rho_{\bar{\alpha}}] = [\bar{\rho}_\alpha]$, and by $[0]$ the class of the identity. Then

$$N_{\alpha\beta}^{\gamma} = N_{\beta\alpha}^{\gamma} = N_{\bar{\alpha}\gamma}^{\beta} = N_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}. \tag{3.13}$$

$$N_0 = \mathbb{1}, \quad N_{\bar{\alpha}} = N_\alpha^T. \tag{3.14}$$

$$N_\alpha N_\beta = N_\beta N_\alpha = \sum_\gamma N_{\alpha\beta}^\gamma N_\gamma. \tag{3.15}$$

$$N_\alpha \cdot \underline{d} = d_\alpha\ \underline{d}, \tag{3.16}$$

where $\underline{d}$ is the vector with components $d_\beta = d(\rho_\beta)$.

4. Markov Traces and Classification

We have in the preceeding section introduced the statistics parameters $\lambda = \omega/d$ as class-invariants of proper representations π_ρ, and class-invariant fusion matrices $N_{\alpha\beta}^{\gamma}$ describing the superselection structure. In general these numbers will not be sufficient as complete characterizations of equivalence classes. But there are huge class-invariant trace states on the commutants $\rho_1 \dots \rho_n(\mathcal{A})'$, which generically encode much more information, and are subject to various functional identities. The latter are very restrictive and (at least in various important cases) can be exploited to obtain a quantization of the admissible values of the parameters. We shall introduce here only the restrictions of these trace states to the images of the braid groups provided by the statistics operators, considered as functionals ("field-theoretical Markov traces") on the braid groups. Under favorite circumstances, these restrictions already uniquely characterize the statistics and the superselection structure.

4.1. Proposition [3,4]: Let ρ be proper, ϕ its left-inverse. The mapping

$$tr_\rho^{(n)} := \phi^n \circ \varepsilon_\rho^{(n)} : B_n \to \mathbb{C} \tag{4.1}$$

extends to a normalized positive strong Markov trace tr_ρ on the group algebra of B_∞; i.e., for $b \in B_n \subset B_{n+1}$ one has

$$tr_\rho^{(n+1)}(b) = tr_\rho^{(n)}(b), \tag{4.2}$$

and the following functional identities hold.

$$tr_\rho(e) = 1, \;\; tr_\rho(b^{-1}) = tr_\rho(b)^*, \tag{4.3}$$

$$tr_\rho(b_1 b_2) = tr_\rho(b_2 b_1). \tag{4.4}$$

If b_1 is a word in the generators $\sigma_1, \dots, \sigma_{k-1}$ and b_2 is a word in the generators $\sigma_k, \dots, \sigma_{n-1}$, $k < n$, then

$$tr_\rho(b_1 b_2) = tr_\rho(b_1)\; tr_\rho(b_2). \tag{4.5}$$

$$tr_\rho(xx^*) \geq 0, \tag{4.6}$$

where $*$ is the anti-linear extension of the inverse on B_∞ to the group algebra.

The trace tr_ρ depends only on the equivalence class of ρ, and

$$tr_{\bar\rho}(\bar b) = tr_\rho(b), \tag{4.7}$$

where if $b = s_1 \dots s_r$ is a word in the symbols $\sigma_i^{\pm 1}$ then $\bar b = s_r \dots s_1$ is the "reversed" word.

The value of tr_ρ on a generator σ_i is the statistics parameter $\lambda(\rho)$.

To give an idea how to exploit this proposition, we shall sketch the quantization of statistics parameters in the "Hecke case" [1,3,5]. The Hecke case is characterized by the validity of a quadratic equation for the statistics operator ε_ρ implying that ε_ρ has at most two different eigenvalues:

$$(\varepsilon_\rho - \mu_1)(\varepsilon_\rho - \mu_2) = 0.$$

This is guaranteed, e.g., in the case of permutation group statistics: $\varepsilon_\rho^2 = 1$, or if ρ^2 has at most two irreducible components: then $\rho^2(\mathcal{A})'$ contains at most two spectral projections of ε_ρ.

Using the quadratic equation in addition to 4.1., one can compute $tr_\rho(b)$ recursively for every braid b, as a function of the parameters $\mu_1, \mu_2, \lambda = tr_\rho(\sigma_1)$. On the other hand, one can find projectors $e_T = e_T^* = e_T^2$ (generalizing Young tableaux of S_n) in the Hecke algebra, which is the group algebra of B_n divided by the ideal generated by $(\sigma_i - \mu_1)(\sigma_i - \mu_2) = 0$. Hence one obtains infinitely many inequalities $tr_\rho(e_T) \geq 0$ from the positivity of the trace. Solving these inequalities for μ_1, μ_2, λ one finds only a discrete series of admissible values for the statistical dimension:

$$d = |\lambda|^{-1} = \frac{\sin \frac{N}{N+L}\pi}{\sin \frac{1}{N+L}\pi},$$

where $N, L = 1, 2, 3, \dots, \infty$ are integers, and relations among the phases. The special case of permutation group statistics, i.e. $\mu_1, \mu_2 = \pm 1$ reduces to $L = \infty$ and

$$d = N = 1, 2, 3, \dots, \quad \omega = \pm 1,$$

where by a strong theorem ([2], see 5.1.), N acquires the interpretation of the dimension of a representation (associated with the sector) of some compact gauge group, and $\omega = \pm 1$ distinguishes bosons from fermions.

A similar analysis has been performed [6] for the case that ρ^2 has three irreducible components one of which is simple. This case reduces to the analysis of traces on Birman-Wenzl algebras.

While 4.1. refers to every sector separately, we want now to turn to a generalization providing more class-invariants (beyond $N^{\gamma}_{\alpha\beta}$) related to the superselection structure.

Let $b \in B_n$, and let $\Lambda : \{1,\ldots,n\} \to \Delta_t$ be a labelling of the symbols $1,\ldots,n$ by transportable endomorphisms. Let π denote the canonical projection of B_n onto S_n, $\pi(\sigma_i) = \tau_i = (i, i+1)$. Putting

$$\begin{aligned} \varepsilon(\Lambda,\sigma_i) &:= \rho_1\ldots\rho_{i-1}(\varepsilon(\rho_i,\rho_{i+1})), \\ \varepsilon(\Lambda,\sigma_i^{-1}) &:= \rho_1\ldots\rho_{i-1}(\varepsilon(\rho_{i+1},\rho_i)^*), \end{aligned} \tag{4.8}$$

where $\rho_j = \Lambda(j)$, the recursive definition of the homomorphism of the groupoid of "colored braids" into the local unitary operators defined by

$$\varepsilon(\Lambda, b_2 b_1) = \varepsilon(\Lambda \circ \pi(b_1)^{-1}, b_2)\varepsilon(\Lambda, b_1) \tag{4.9}$$

is unambiguous by virtue of 2.1., and with $\rho_j = \Lambda(j)$ and $\pi = \pi(b)$,

$$\varepsilon(\Lambda, b) \in (\rho_{\pi^{-1}(1)}\ldots\rho_{\pi^{-1}(n)}|\rho_1\ldots\rho_n). \tag{4.10}$$

Considering braids as topological objects, this definition amounts to the choice of a two-dimensional projection of the braid, assigning "colors" ρ_j to its lines, and representing every vertex by the appropriate statistics operator of the two colors involved.

In this picture, applying left-inverses can be interpreted as "closing" the braid into a link by identifying its in- and out-going lines. Hence one is led to consider pairs (Λ, b) such that $\Lambda \circ \pi(b) = \Lambda$. These are precisely those pairs, for which $\varepsilon(\Lambda, b)$ lie in the commutant $\rho_1\ldots\rho_n(\mathcal{A})'$, i.e., those relevant for the study of the reducibility of composite representations. Equivalently one can say that Λ is constant on the cycles $(i, \pi(i), \ldots, \pi^{l-1}(i))$ of $\pi = \pi(b)$, which in turn correspond to the components of the link obtained by closing the braid. We shall refer to the cycles of $\pi(b)$ as the components of b, and call (Λ, b) a labelling of the components of b if $\Lambda \circ \pi(b) = \Lambda$.

In order to be able to apply left-inverses, we have to restrict Λ to take values which are products of proper endomorphisms $\rho_\alpha\ldots\rho_\gamma \in \Delta_t$. Denote by L_n the set of all labellings of the components of braids in B_n by products of proper endomorphisms.

4.2. Proposition: Define the functional tr on L_n by

$$tr(\Lambda, b) := \phi_n\ldots\phi_1(\varepsilon(\Lambda, b)) \in \mathbb{C}, \tag{4.11}$$

where if $\rho_i \equiv \Lambda(i) = \rho_\alpha\ldots\rho_\gamma$ then $\phi_i = \phi_\gamma\ldots\phi_\alpha$ is the "standard left-inverse" of ρ_i, which is the product of the unique left-inverses of the factors. Then tr is a class-invariant functional, i.e. its values are independent of the choice of representatives $\rho_\alpha \in [\rho_\alpha]$ of the proper endomorphisms involved. It satisfies the following functional equations.

If $b \in B_n$ is identified with $b' = b \in B_{n+1}$, and if $\Lambda'(i) = \Lambda(i)$ for $i \le n$, then, independently of the label $\Lambda'(n+1)$,

$$tr(\Lambda', b') = tr(\Lambda, b). \tag{4.12}$$

$$tr(\Lambda, e) = 1, \quad tr(\hat{\Lambda}, b^{-1}) = tr(\Lambda, b)^*. \tag{4.13}$$

$$\begin{aligned} tr(\Lambda, b_1 b_2) &= tr(\Lambda \circ \pi_2^{-1}, b_2 b_1) \\ &\equiv tr(\Lambda \circ \pi_1, b_2 b_1), \end{aligned} \tag{4.14}$$

where $\pi_i = \pi(b_i)$.

If b_1 is a word in the generators $\sigma_1,\ldots,\sigma_{k-1}$ and b_2 is a word in the generators $\sigma_k,\ldots,\sigma_{n-1}$, $k < n$, and if $\rho_k = \Lambda(k)$ is irreducible, then

$$tr(\Lambda, b_1 b_2) = tr(\Lambda, b_1)\cdot tr(\Lambda, b_2). \tag{4.15}$$

If Λ is the constant labelling $\Lambda(i) = \rho$, ρ proper, then

$$tr(\Lambda, b) = tr_\rho(b). \tag{4.16}$$

Putting $\bar{b}$ as in 4.1., and $\bar{\Lambda}(i) = \bar{\rho}_\gamma\ldots\bar{\rho}_\alpha$ if $\Lambda(i) = \rho_\alpha\ldots\rho_\gamma$, then

$$tr(\Lambda, b) = tr(\bar{\Lambda}, \bar{b}). \tag{4.17}$$

If some cycle $(i, \pi(i), \ldots)$ of length l is labelled by $\Lambda(i) = \Lambda(\pi(i)) = \ldots = \rho_\alpha\rho_\beta$, then

$$tr(\Lambda, b) = \sum_{\gamma\in\nabla} N^{\gamma}_{\alpha\beta}\left[\frac{d_\gamma}{d_\alpha d_\beta}\right]^l tr(\Lambda_\gamma, b), \tag{4.18}$$

where $\Lambda_\gamma(i) = \Lambda_\gamma(\pi(i)) = \ldots = \rho_\gamma \in [\rho_\gamma]$, and $\Lambda_\gamma(j) = \Lambda(j)$ else.

For $\Lambda(1) = \rho_\alpha$, $\Lambda(2) = \rho_\beta$, and $\Lambda'(1) = \rho_\alpha$, $\Lambda'(2) = \bar{\rho}_\beta$ one has

$$tr(\Lambda, \sigma_1^{-2}) = tr(\Lambda', \sigma_1^{+2}). \tag{4.19}$$

(Except for the last two statements, this is a rather straightforward generalization of the previous proposition. (4.18) and (4.19) are proven by repeated use of (2.3) and Theorem 3.2.)

Observe that if one expands $\varepsilon(\Lambda,\sigma_i) = \varepsilon(\rho_\alpha \dots \rho_\gamma, \rho_\beta \dots \rho_\delta)$ by virtue of (2.5) into products of statistics operators of the involved proper factors, and similarly $\varepsilon(\Lambda,\sigma_i^{-1})$, one finds that every $\varepsilon(\Lambda,b)$ is rewritten in the form $\varepsilon(\tilde\Lambda,\tilde b)$, where $\tilde b \in B_N$ is an "amplified" braid for some $N \geq n$ and $\tilde\Lambda$ is a labelling of the components of $\tilde b$ by proper endomorphisms ($(\tilde\Lambda,\tilde b)$ is obtained from (Λ,b) by replacing every line (i) of the braid by several lines labelled with the proper factors of $\Lambda(i)$), and $tr(\tilde\Lambda,\tilde b) = tr(\Lambda,b)$. In fact, the above functional identities follow from the same identities among $tr(\Lambda,b)$ such that $\Lambda(i)$ are all proper, if the left-hand-side of (4.18) is understood in the described identification.

Let us now evaluate some of the restrictions expressed in 4.2. [7]. Define the matrix $Y_{\alpha\beta}$, $\alpha,\beta \in \nabla$:

$$Y_{\alpha\beta} := d_\alpha d_\beta\, tr(\Lambda,\sigma_1^{-2}), \tag{4.20}$$

where $\Lambda(1) \in [\rho_\alpha]$, $\Lambda(2) \in [\rho_\beta]$. The trace and conjugation properties (4.14), (4.17) and (4.19) imply

$$Y_{\alpha\beta} = Y_{\beta\alpha} = Y_{\bar\alpha\bar\beta} = Y^*_{\alpha\bar\beta}, \tag{4.21}$$

The fusion property (4.18) with $b = \sigma_1^{-1}$, $\Lambda(1) = \Lambda(2) = \rho_\alpha\rho_\beta$, and the factorization property (4.15) yield

$$Y_{\alpha\beta} = \sum_\gamma N^\gamma_{\alpha\beta} \frac{\omega_\alpha\omega_\beta}{\omega_\gamma} d_\gamma. \tag{4.22}$$

Again (4.15) with $b = \sigma_1^{-2}$, $\Lambda(1) = \rho_\alpha\rho_\beta$, $\Lambda(2) = \rho_\delta$ yields

$$d_\delta^{-1}\, Y_{\alpha\delta} Y_{\beta\delta} = \sum_\gamma N^\gamma_{\alpha\beta}\, Y_{\gamma\delta}. \tag{4.23}$$

These equations imply the following [7, Sect.5].

4.3. Proposition: Assume that ∇ (or some subset of proper sectors closed under composition with subsequent reduction, and conjugation) is finite, hence Y a finite matrix. Then:

Either Y is invertible. Putting $\sigma := \sum_\alpha d_\alpha^2 \omega_\alpha^{-1}$, and

$$S := |\sigma|^{-1}\, Y, \qquad T := (\sigma/|\sigma|)^{1/3} Diag(\omega_\alpha), \tag{4.24}$$

one has $|\sigma|^2 = \sum_\alpha d_\alpha^2$, and

$$SS^\dagger = TT^\dagger = \mathbb{1}, \quad TSTST = S, \quad S^2 = C, \;\; TC = CT = T, \tag{4.25}$$

where $C_{\alpha\beta} = \delta_{\bar\alpha\beta}$ is the conjugation matrix, and

$$N^\gamma_{\alpha\beta} = \sum_\delta \frac{S_{\alpha\delta} S_{\beta\delta} S^*_{\gamma\delta}}{S_{0\delta}}. \tag{4.26}$$

(This algebra is famous from "rational" conformal field theories [8], but is seen here not to depend on any covariance or modular properties.)

Or Y is degenerate. Then there is a non-trivial subset $\nabla_{\rm deg}$ of "degenerate" sectors $[\gamma]$ such that $Y_{\gamma\delta} = d_\gamma d_\delta$ for all $\delta \in \nabla$, and all other sectors may be grouped into "families", such that for $[\rho_\alpha], [\rho_\beta]$ in the same family the corresponding column vectors of the matrix Y are parallel, and $\omega_\alpha = \pm\omega_\beta$. For $[\rho_\alpha], [\rho_\beta]$ in different families, the corresponding column vectors of the matrix Y are orthogonal. The degenerate sectors have permutation group statistics, are closed under composition with subsequent reduction and conjugation, and act irreducibly within every family (i.e. ρ_α, ρ_β belong to the same family if and only if there is γ belonging to a degenerate sector $[\gamma]$ such that $N^\beta_{\gamma\alpha} \neq 0$. γ is bosonic if $\omega_\alpha = \omega_\beta$, and fermionic otherwise.)

A proper endomorphism γ belongs to a degenerate sector, if and only if its monodromy with every proper ρ is trivial:

$$\varepsilon(\gamma,\rho)\varepsilon(\rho,\gamma) = 1. \tag{4.27}$$

The two alternatives of 4.3. give some insight into the general superselection structure of theories with braid group statistics. The first one raises the challenging question to classify the solutions to the "self-dual" structure (4.25), (4.26) and the related "hypergroup" of matrices (3.13–16), and possibly to understand it in terms of some "quantum" symmetry. Many examples are known, e.g., in connection with conformal [8] and lattice integrable [9] models. The second one suggests [7] that there is a maximal extension of the local algebra by fields carrying the degenerate charges, such that the superselection structure of the extended algebra is given by the first alternative. We shall elaborate more on this dichotomy of general braid group statistics into a permutation group statistics part associated with an ordinary symmetry group, and a self-dual part, in the following section.

5. Algebra Extensions

Let us first discuss the case of pure permutation group statistics, which is equivalent to all sectors being degenerate. This case is in particular relevant for four-dimensional theories, and establishes that the superselection structure can be deduced from the action of a global gauge group.

5.1. Theorem [2]: Assume that ∇ has pure permutation group statistics, i.e.

$$\varepsilon(\gamma_1,\gamma_2)\varepsilon(\gamma_2,\gamma_1) = 1 \quad \forall\ \gamma_i \in [\gamma_i] \in \nabla.$$

Then there is a $\mathbb{Z}_2$-graded local algebra $\mathcal{F} \supset \mathcal{A}$ with trivial center, generated by $\mathcal{A}$ and a collection of isometric operator multiplets (the charged fields) $\psi_i^{(\gamma)}$, $\gamma \in [\gamma] \in \nabla$, $i = 1,\dots,d(\gamma)$:

$$\psi_i^{(\gamma)*}\psi_j^{(\gamma)} = \delta_{ij}. \tag{5.1}$$

The latter create the superselection charges of $\mathcal{A}$:

$$\psi_i^{(\gamma)}A = \gamma(A)\psi_i^{(\gamma)}. \tag{5.2}$$

There is a compact global gauge group G of automorphisms of $\mathcal{F}$, unitarily and irreducibly acting on the internal degrees of freedom i, such that the equivalence classes of irreducible representations of G are in 1:1 correspondence with ∇ and have the same composition rules. $\mathcal{A}$ is the fixpoint subalgebra of $\mathcal{F}$. In particular,

$$\sum_i \psi_i^{(\gamma)}\psi_i^{(\gamma)*} = 1, \tag{5.3}$$

$$\pm\sum_{ij}\psi_i^{(\gamma_2)}\psi_j^{(\gamma_1)}\psi_i^{(\gamma_2)*}\psi_j^{(\gamma_1)*} = \varepsilon(\gamma_1,\gamma_2), \tag{5.4}$$

with the $-$ sign if both γ_1,γ_2 are fermionic. The subalgebras $\mathcal{F}(\mathcal{O})$ are generated by $\mathcal{A}(\mathcal{O})$ and $\psi_i^{(\gamma)}$, γ localized in $\mathcal{O}$. Then, if γ_1,γ_2 are localized in $\mathcal{O}_1 \times \mathcal{O}_2$, the normal commutation relations hold:

$$\psi_i^{(\gamma_1)}\psi_j^{(\gamma_2)} = \pm\psi_j^{(\gamma_2)}\psi_i^{(\gamma_1)}. \tag{5.5}$$

Up to a phase, one has

$$U\psi_i^{(\gamma)} \propto \psi_i^{(\sigma_U\circ\gamma)}. \tag{5.6}$$

The vacuum representation π_0 of $\mathcal{F}$, defined as the GNS-construction from the state functional

$$\Omega_0(A\psi_i^{(\gamma)}) := \delta_{[\gamma][id]}\,\langle\Omega,\pi_0(A\psi_i^{(\gamma)})\Omega\rangle, \tag{5.7}$$

where Ω is the vacuum state vector of $\mathcal{A}$, contains all sectors $[\gamma]$ of $\mathcal{A}$ with multiplicity $d(\gamma)$.

Now consider the general situation with braid group statistics occurring. Let $\nabla_{\rm pgs} \subset \nabla$ be a closed subset of bosonic sectors with permutation group statistics (the corresponding proper endomorphisms are generically denoted γ), e.g., the subset $\nabla_{\rm deg+}$ of bosonic degenerate sectors in the second alternative of 4.3. The DR-construction 5.1. applies to $\nabla_{\rm pgs}$. Let $\mathcal{F}$ be the corresponding field algebra. (The restriction to the bosonic case is for the sake of simplicity.)

5.2. Proposition: If and only if a proper endomorphism ρ of $\mathcal{A}$ satisfies (4.27) for all $\gamma \in [\gamma] \in \nabla_{\rm pgs}$, it extends uniquely to a localized and transportable endomorphism (also called ρ) of $\mathcal{F}$ by

$$\rho(\psi_i^{(\gamma)}) = \varepsilon(\rho,\gamma)^*\psi_i^{(\gamma)}. \tag{5.8}$$

The extension is in general reducible. Its irreducible components $\rho^{(j)}$ are proper endomorphisms of $\mathcal{F}$, such that their statistical dimensions satisfy the sum rule

$$\sum_j d(\rho^{(j)}) = d(\rho), \tag{5.9}$$

and their statistics phases coincide:

$$\omega(\rho^{(j)}) = \omega(\rho). \tag{5.10}$$

In the case $\nabla_{\rm pgs} = \nabla_{\rm deg+}$: if and only if ρ_α,ρ_β belong to the same family (in the sense of 4.3.) and $\omega_\alpha = \omega_\beta$, their extensions possess common subrepresentations.

Sketch of a proof: Requiring triviality of ρ (localized in $\mathcal{O}$) on $\mathcal{F}(\mathcal{O}^-)$ determines (5.8) (using (5.6) and (2.3–4)), while triviality on $\mathcal{F}(\mathcal{O}^+)$ determines

$$\rho(\psi_i^{(\gamma)}) = \varepsilon(\gamma,\rho)\psi_i^{(\gamma)}.$$

The compatibility is precisely (4.27). Next, the commutant $\rho(\mathcal{F})'$ is the finite-dimensional vonNeumann algebra spanned by the intertwiners

$$\psi_i^{(\gamma)*}T, \quad \begin{cases} T \in (\gamma\rho|\rho) \subset \mathcal{A}, \\ \gamma \text{ bosonic degenerate.} \end{cases}$$

Its decomposition into matrix rings, and hence the reducibility of the extension of ρ, is governed by the multiplication law

$$\psi_{i_1}^{(\gamma_1)*}T_1 \cdot \psi_{i_2}^{(\gamma_2)*}T_2 =$$
$$\sum_{\gamma,j}(\psi_{i_1}^{(\gamma_1)*}\psi_{i_2}^{(\gamma_2)*}T_g\psi_j^{(\gamma)})\ \psi_j^{(\gamma)*}T\ (T^*T_g^*\gamma_2(T_1)T_2),$$

where $T_g \in (\gamma_2\gamma_1|\gamma)$, $T \in (\gamma\rho|\rho)$ are orthonormal bases of isometries, and the operators in brackets are actually scalars: the first ones lie in the center of $\mathcal{F}$ and coincide with the Clebsch-Gordan coefficients of the group G; the second ones lie in the center of $\mathcal{A}$ and coincide with the operator product structure constants D of the "reduced field bundle" [7]. Similarly, in the case $\nabla_{\text{pgs}} = \nabla_{\text{deg}+}$, the space of intertwiners in $\mathcal{F}$ from ρ_β to ρ_α is spanned by $\psi_i^{(\gamma)*}T$ with $T \in (\gamma\rho_\alpha|\rho_\beta)$, γ bosonic degenerate, which by 4.3. implies the last statement. The remaining formulae are derived by standard techniques [1,3,4] from the observation, that left-inverses of proper endomorphisms of $\mathcal{A}$ extend to standard left-inverses of their extensions to $\mathcal{F}$.

It is instructive to consider an example. (The reader familiar with conformal models will recognize the extension of the level-4 SU(2) current algebra $\mathcal{A}$ to the level-1 SU(3) current algebra $\mathcal{F}$ [10]). Let ∇ consist of three self-conjugate sectors [0], [1], [2] with fusion rules

$$N_{0i}^j = \delta_{ij}, \quad N_{22}^j = \delta_{j0}, \quad N_{12}^j = \delta_{j1}, \quad N_{11}^j = 1.$$

These determine the statistical dimensions to be $d(0) = d(2) = 1, d(1) = 2$. Assuming the sector [2] to be bosonic degenerate, and [1] non-degenerate, (4.21–23) determine the matrix Y

$$Y_{\mathcal{A}} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{pmatrix},$$

and $\omega = \omega(1)$ to be a non-trivial third root of unity. By the DR-construction 5.1., $\mathcal{F}$ is generated by $\mathcal{A}$ and one unitary charged operator ψ, such that

$$\gamma(F) = \psi F \psi^*$$

restricts to an automorphism $\gamma \in [2]$ of $\mathcal{A}$. The group $G = \mathbb{Z}_2$ acts on $\mathcal{F}$ by $g(\psi) = -\psi$, $g(A) = A$.

Let $\rho \in [1]$, $T \in (\gamma\rho|\rho) \subset \mathcal{A}$. Then $\rho(\mathcal{F})'$ is spanned by 1 and ψ^*T, and the latter may be normalized such that $(\psi^*T)^2 = 1$. Hence there are precisely two projections

$$E^\pm = \frac{1}{2}(1 \pm \psi^*T),$$

hence two proper components $\rho^\pm$ of the extension of ρ, which have both $d(\rho^\pm) = 1$, hence are automorphisms of $\mathcal{F}$ by 3.3. Since ρ was self-conjugate, either $\rho^\pm$ are self-conjugate or conjugate to each other. The first possibility is ruled out, since self-conjugate automorphisms have $\omega^4 = 1$ [7, Sect.3]. Hence the superselection structure of $\mathcal{F}$ is given by the matrix

$$Y_{\mathcal{F}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

the character table of $\mathbb{Z}_3$ satisfying the first alternative of 4.3. This and other examples give further evidence to the conjecture [7] that for $\nabla_{\text{pgs}} = \nabla_{\text{deg}}$, $Y_{\mathcal{F}}$ is always nondegenerate. We hope to return to this issue elsewhere.

If in 5.2., ∇_{pgs} is larger than $\nabla_{\text{deg}+}$, then there are endomorphisms ρ of $\mathcal{A}$ not satisfying (4.27) for all $\gamma \in [\gamma] \in \nabla_{\text{pgs}}$. These can still be extended to endomorphisms of $\mathcal{F}$, but the extensions will no longer be trivial on *both* $\mathcal{F}(\mathcal{O}^-)$ and $\mathcal{F}(\mathcal{O}^+)$. However, $\Omega_0 \circ \rho$ and Ω_0 still coincide on both causal complements, showing that $\pi_0 \circ \rho$ is still equivalent to – but cannot be simultaneously put equal to π_0 in restriction to both causal complements. These are special cases of *soliton representations* [11] of $\mathcal{F}$. Soliton representations may by definition even approach inequivalent vacua on both sides, and should not be excluded *a priori* in low dimensions. In fact, although we have restricted ourselves to localized endomorphisms, we see the solitonic ones naturally re-emerge.

If either ∇_{pgs} consists of simple sectors only (hence the gauge group G is abelian), or if the endomorphism ρ to be extended is simple, then the monodromy factors are scalars, and ρ, chosen to be trivial on $\mathcal{F}(\mathcal{O}^-)$, acts on $\mathcal{F}(\mathcal{O}^+)$ as an abelian group of automorphisms. (The selection rules for statistics phases [7]

$$\Omega_{\tau_1}(\rho)\Omega_{\tau_2}(\rho) = \Omega_{\tau_1\tau_2}(\rho),$$
$$\Omega_\tau(\rho_1)\Omega_\tau(\rho_2) = \Omega_\tau(\rho_3) \quad \text{if} \quad (\rho_1\rho_2|\rho_3) \neq \{0\},$$

where τ are automorphisms, ρ are proper, and $\Omega_\tau(\rho) := \varepsilon(\rho,\tau)\varepsilon(\tau,\rho) = \omega(\tau\rho)/\omega(\tau)\omega(\rho)$ are scalar phases, turn out to be the product and the coproduct structures of this group.) The reader familiar with orbifold models of conformal field theory [12] will recognize the "twisted sectors" here. In fact, if we return to our example above and restrict the algebra $\mathcal{F}$ to its subalgebra $\mathcal{A}$, we find that $\mathcal{A}$ possesses two more sectors $[\frac{1}{2}], [\frac{3}{2}]$ of localized representations, with respect to which γ is non-degenerate:

$$\varepsilon(\gamma,\rho_i)\varepsilon(\rho_i,\gamma) = -1, \quad \rho_i \in [\tfrac{1}{2}], [\tfrac{3}{2}],$$

and which consequently come only from (or: can only be extended to) a solitonic representation of $\mathcal{F}$.

In general, however, if neither ρ nor γ is simple, then the action of ρ on $\psi_i^{(\gamma)} \in \mathcal{F}(\mathcal{O}^+)$ (the "twist") cannot be considered as a group action and does not even leave $\mathcal{F}(\mathcal{O}^+)$ invariant. Many results of the previous sections, starting with (2.1), will fail as they stand, for general solitonic endomorphisms of $\mathcal{F}$.

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