Braid Group Statistics and their Superselection Rules

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Abstract: We present recent results on the statistics in low-dimensional quantum field theory. They are described by unitary representations of the braid group. We discuss the structure of the "reduced field bundle" which is a charged field algebra exhibiting the braid group in its commutation relations ("exchange algebra"). We systematize results about the superselection rules for sectors with braid group statistics.

1 Introduction

To classify quantum field theories is a very elusory task. The choice of a framework (Lagrangean, Wightman, ...) may already, and in a most uncontrolled manner, restrict the class of theories in the competition. Famous "no go" theorems may fail if one changes the framework; just think of the new prospects hopefully expected from string theories.

In particular, the description of a physical system in terms of unobservable fields carrying superselection charge is highly ambiguous (bosonization, Klein transformations, Bogoliubov transformations, ...). An unbiased approach to quantum field theory should be based on "first principles" to the largest possible extent, and should attempt to avoid the *ad hoc* introduction of charged fields, (but certainly be capable to describe charged states!). One such approach is the framework of algebraic quantum field theory [1], developped to great perfection [2, 3, 4] but unfortunately widely ignored. This is partially due to the criticism that it does not make specific dynamical predictions; but on the contrary, it owes to its independence of model assumptions a great conceptual clearness, separating the peculiarities of a model from the characteristics of quantum field theory. Moreover, its affirmative value of justifying and refining our understanding of the structure of particles (localizability [5], existence of anti-particles [6]) and symmetry [7] can hardly be over-estimated.

The analysis of superselection sectors and statistics as an intrinsic characterization of C^* algebras (local nets) of observables [3, 4, 8, 9] is another example for the power of the algebraic approach. The prominent result is that in sufficiently high ($\geq 2+1$ or $\geq 3+1$ depending on the localizability of charges) space-time dimensions, statistics are permutation group statistics, while below they are braid group statistics. While the former have been completely classified (Fermi or Bose para-statistics), the latter are only partially classified; but the *invariant* (description independent) information about braid group statistics is encoded in positive Markov traces associated with superselection charges, which are accessible to classification by powerful mathematical tools [10].

The detailed study of braid group statistics leads to a natural (though not canonical) construction of a charged field algebra, the "reduced field bundle" $\mathcal{F}$ [8, 9], *from the observable content*, i.e. the algebra of observables $\mathcal{A}$, of the theory. This structure precisely predicts all the remarkable structural observations, made in large classes of two-dimensional models of conformal field theory and commonly ascribed to the peculiarities of conformal invariance; but in fact, the abstract derivation is completely model independent and in particular has never to assume conformal invariance. These issues, which have been worked out in collaboration with K.Fredenhagen and B.Schroer, are covered in Sect.2 of this contribution.

As a matter of fact, the conformal models appear quite exhaustive for the admissible braid group statistics, as far as the latter are pinned down, in the crude sense that every complication that cannot be ruled out by general arguments is indeed realized in some of these models. One might speculate whether the conformal models provide some "complete sample collection of prototypes" of braid group statistics, in this respect comparable to the free theories scanning permutation group para-statistics. But whatever the situation is, braid group statistics being the natural statistics in low dimensions, one should expect, e.g., massive two-dimensional theories (or three-dimensional gauge theories) with non-abelian braid group statistics to exist, which we do not yet know by "lack of phantasy" in model building.

The braid group statistics underlying two-dimensional conformal models are in fact the statistics of one-dimensional light-cone theories. By virtue of conformal covariance, space-time fields factorize as bilinear expressions in light-cone fields. The role of the center of the conformal covariance group in this decomposition, related to the solution of the "causality paradox" which appears at the transition from Euclidean to relativistic conformal field theories, has been emphasized very early by J.A.Swieca and his collaborators [11], while the discreteness of its spectrum was touched upon by Lüscher and Mack [12]. The rediscovery of these crucial issues in a different framework based on the analyticity properties of Euclidean correlation functions [13] initiated the enormous recent progress in conformal field theory. Interpreting conformal block functions as vacuum expectation values of light-cone fields interpolating different superselection sectors, their monodromy properties turn into commutation relations (exchange algebra) of the latter [14, 15]. By a non-trivial interplay of the structure constants of the two exchange algebras (on either light-cone), this decomposition is compatible with local commutativity, or more general "conventional" commutation relations (as opposed to exchange algebra commutation relations with structure constants depending on the charges among which the operators interpolate), of the space-time fields [16, 17, 18].

For permutation group statistics it has been established [19] that the irreducible sectors are in one-to-one correspondence with the representations of some compact group, and that the selection rules for the composition and reduction of sectors coincide with the duality theory of this group. The role of the compact group is that of a global gauge symmetry; in fact there exists a field algebra [7] with a linear action of the symmetry group, the invariant subalgebra of which coincides with the algebra of observables, while charged (non-invariant) operators of the field algebra implement the non-trivial superselection sectors.

The analogue of this symmetry structure for sectors with braid group statistics (e.g., [20, 21]) is not yet completely understood. In many examples of conformal field theories, one encounters the representation theory of "quantum groups" [22] at the singular values of their parameter q, and various attempts have been made to understand the action of these objects on field operators. But in the general case, the situation remains quite unclear. In particular, one has few control over the superselection rules for sectors with braid group statistics, and neither are the composition rules for fractional spins, related to the statistics phases, understood. It is the major aim of the present contribution to systematize various partial results on this issue.

Some interesting results can be derived for abelian sectors (outer automorphisms of the local algebra), and the selection rules for their composition with generic sectors; these will be presented in Sect.3.

We shall point out the role of the Markov trace for the classification of statistics of a single sector, and of the interplay of different sectors. In the special case of a sector whose square contains only two inequivalent subsectors, the Markov trace is a positive trace on the Hecke algebra. The classification of positive Markov traces on the Hecke algebra leads to a quantization of the possible statistics of such a sector, and provides detailed results about the other sectors generated by it. A similar analysis seems possible for self-conjugate three-channel sectors [23], where one has to deal with the Birman-Wenzl algebra [24]. These issues are treated in Sect.4.

In Sect.5 we discuss a "global" issue: the fascinating interplay of *all* sectors of a theory. Every sector is assigned a vector in some "weight space", defined in terms of statistics operators and left-inverses. The significance of the weight vectors, which after some metric re-normalization we call "statistics characters" by analogy, generalizes that of the characters of finite symmetry groups. The statistics characters diagonalize the fusion rules with a duality between eigenvectors and eigenvalues. Any two weight vectors are either orthogonal or parallel; in particular, the weight vector of the vacuum sector being the characteristic direction of sectors with permutation group statistics, one finds that the latter cannot be "continuously" approached by braid group statistics. If the theory contains only sectors with true braid group statistics, then the matrix of weight vectors is unitary (up to a factor) and satisfies a remarkable algebra together with the diagonal matrix of statistics phases. In conformal models, this algebra coincides with the well-known modular transformation algebra for Virasoro characters of the local algebra [25, 26], but its occurrence in the general case remains a mystery. In the other extremal case of a theory with permutation group statistics only, the matrix of statistics characters becomes completely redundant and should be substituted by the character table of the symmetry group.

We shall include proofs in the present contribution only for new results. For details about well-known results we refer the reader to the original literature, or to a pedagogical introduction [27].

2 Statistics and the Reduced Field Bundle

In this section we review the results of [8, 9] pertaining to the "reduced field bundle".

Superselection sectors associated with localizable charges are most conveniently described by C^* morphisms ρ of the algebra of observables $\mathcal{A}$. Identifying $\mathcal{A}$ with its vacuum representation, charged sectors are thus represented in the vacuum Hilbert space, with observables acting *via the morphism*:

$$A : \mathcal{H}_\rho \ni (\rho, \Psi) \mapsto (\rho, \rho(A)\Psi) \in \mathcal{H}_\rho, \tag{2.1}$$

where $\mathcal{H}_\rho = \mathcal{H}_0$ as a vector space, $\Psi \in \mathcal{H}_0$, and the "crossed product" notation (ρ, Ψ) is used to indicate the nontrivial action of $\mathcal{A}$. We denote by $(\rho_2|\rho_1)$ the set of operators intertwining from ρ_1 to ρ_2 (in the sense of the actions (2.1)).

The statistics of a sector is a unitary operator $\varepsilon_\rho = \varepsilon(\rho, \rho) \in \rho^2(\mathcal{A})'$ inducing a representation of the infinite permutation group (in high dimensions) or braid group (in low dimensions), see Sect. 4. More generally, for any two sectors there are unitary intertwiners $\varepsilon(\rho_1, \rho_2)$ from $\rho_1\rho_2$ to $\rho_2\rho_1$ given by the following

Definition and Proposition: Let ρ_i be localized in $\mathcal{O}_i$. Let $\hat\rho_i \in [\rho_i]$ be localized in $\hat{\mathcal{O}}_i$ such that $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$ are at space-like distance, and $U_i \in (\hat\rho_i|\rho_i)$ unitary "charge transporters". Then the unitary statistics operator $\rho_2(U_1^*)U_2^*U_1\rho_1(U_2)$ is independent of U_i and does not change if $\hat{\mathcal{O}}_i$ are continuously changed within the space-like complements of each other. Thus, in dimension $d \leq 1+1$, where $\mathcal{O}'$ has two connected components, it can take only two values:

$$\rho_2(U_1^*)U_2^*U_1\rho_1(U_2) =: \begin{cases} \varepsilon(\rho_1,\rho_2) & \text{if} \quad \hat{\mathcal{O}}_2 < \hat{\mathcal{O}}_1 \\ \varepsilon(\rho_2,\rho_1)^* & \text{if} \quad \hat{\mathcal{O}}_1 < \hat{\mathcal{O}}_2 \end{cases} \tag{2.2}$$

where some space-like ordering $<$ has been chosen. In $d \geq 2+1$ these two values coincide. The following identities hold:

$$\rho_3(\varepsilon(\rho_1,\rho_2))\varepsilon(\rho_1,\rho_3)\rho_1(\varepsilon(\rho_2,\rho_3)) = \varepsilon(\rho_2,\rho_3)\rho_2(\varepsilon(\rho_1,\rho_3))\varepsilon(\rho_1,\rho_2), \tag{2.3}$$

$$\rho(T)\varepsilon(\rho_3,\rho) = \varepsilon(\rho_1\rho_2,\rho)T = \varepsilon(\rho_1,\rho)\rho_1(\varepsilon(\rho_2,\rho))T \tag{2.4}$$

for $T \in (\rho_1\rho_2|\rho_3)$, and similar with $\varepsilon(\rho,\rho')$ replaced by $\varepsilon(\rho',\rho)^*$ everywhere.

Definition: For ρ irreducible, $\bar{\rho}$ a conjugate, $R \in (\bar{\rho}\rho|id)$ an isometry, the statistics parameter of the sector $[\rho]$ is

$$\lambda_\rho := R^*\bar{\rho}(\varepsilon_\rho)R \in \mathbb{C}. \tag{2.5}$$

The statistics parameter, as an element of $(\rho|\rho) = \rho(\mathcal{A})'$, is a scalar. It is independent of the choice of the isometry R and depends only on the equivalence class of ρ. The statistics parameters of conjugate morphisms coincide. We call the statistics non-degenerate or finite, if $\lambda(\rho) \neq 0$. Then we denote by

$$\frac{\omega(\rho)}{d(\rho)} := \lambda(\rho) = \lambda(\bar{\rho}) \tag{2.6}$$

the polar decomposition into a phase $\omega(\rho)$ (statistics phase) generalizing the distinction between bosons and fermions, and the inverse modulus $d(\rho) \geq 1$ (statistical dimension) generalizing the order of (permutation group) para-statistics.

Spin-Statistics Theorem [9]: For conformally covariant theories on the light-cone,

$$\omega(\rho) = \exp\ 2\pi i\ h_\rho, \tag{2.7}$$

relates the statistics phase $\omega(\rho)$ of a covariant sector to the conformal scaling dimensions $h_\rho (mod\ \mathbb{Z})$ of fields carrying charge $[\rho]$.

Remarks: (1) Analogues are expected to hold also for the Poincaré spin of more general low-dimensional exchange fields. So far, however, the validity of a Spin-Statistics theorem is established only for covariance groups that can geometrically change the sign of a space-like separation by real transformations: the conformal group acting on the compactified light-cone, and the Poincaré group in $2+1$ dimensions. In fact, this action of the covariance group does not imply that the two statistics operators (2.2) coincide, since in these situations the relevant ordering is defined with respect to some reference frame (a "point at infinity" [9] resp. a space-like direction [28]), but rather relates their difference (the "monodromy" operator $\varepsilon(\rho_1,\rho_2)\varepsilon(\rho_2,\rho_1)$)

to the covariance quantum numbers (spin).
(2) In conformal models, the statistical dimensions $d(\rho)$ are known as the normalized entries $S_{0\rho}/S_{00}$ of the modular matrix, measuring the relative dimensions of representations of the chiral algebra [25, 26]. In general (confirming an old idea of S.Doplicher), it measures the index $Ind = d(\rho)^2$ of the inclusion of vonNeumann factors $\rho(\mathcal{A}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O})$ [23].

Before we introduce the "reduced field bundle", let us physically motivate the abstract action of charged fields interpolating among different superselection sectors.

Let $\rho, \rho_\alpha, \rho_\beta$ be irreducible transportable morphisms such that ρ_β is equivalent to some subrepresentation of $\rho_\alpha\rho$. Let $T_e \in (\rho_\alpha\rho|\rho_\beta)$ be an isometry. For $A \in \mathcal{A}$ define the linear operator $(e, A) : \mathcal{H}_{\rho_\alpha} \to \mathcal{H}_{\rho_\beta}$ by

$$(e, A)\ (\rho_\alpha, \Psi) := (\rho_\beta, T_e^* \rho_\alpha(A)\Psi). \tag{2.8}$$

This corresponds to the action of A in the background charge ρ_α, addition of the charge ρ, and subsequent projection and unitary transport by means of $T_e^* \in (\rho_\beta|\rho_\alpha\rho)$ of the state $(\rho_\alpha\rho, \rho_\alpha(A)\Psi) \in \mathcal{H}_{\rho_\alpha\rho}$ to a state in $\mathcal{H}_{\rho_\beta}$. The collective label e ("superselection channel") stands for the three irreducible morphisms involved as well as for the specific intertwiner T_e chosen, see below. We shall call $s(e) = \rho_\alpha, r(e) = \rho_\beta$ the "source" and the "range" of e (referring to the interpolation of the map (e, A)), and $c(e) = \rho$ the "charge" of e (referring to the charge added by the operator (e, A)), and write $e = (\rho_\alpha, \rho, \rho_\beta)$ if we want to specify only its source, charge, and range.

We shall now restrict ourselves to the set Δ_0 of transportable morphisms, possessing conjugates and having finite statistics. Δ_0 is closed under composition and taking subrepresentations [3, 4]. Moreover, if $\rho_\alpha, \rho \in \Delta_0$ are irreducible, then $\rho_\alpha\rho$ contains only finitely many inequivalent irreducible subrepresentations ρ_β, each occurring with finite multiplicity

$$(N_\rho)_\alpha^\beta \equiv \dim(\rho_\alpha\rho|\rho_\beta) < \infty. \tag{2.9}$$

Definition: Let $\nabla_0 \subset \Delta_0$ be a countable collection of "reference" morphisms, one per equivalence class of irreducible morphisms in Δ_0 (or in some subset Δ closed under composition and taking subrepresentations and conjugates), $id \in \nabla_0$. For every triple $\rho, \rho_\alpha, \rho_\beta \in \nabla_0$ let $N = (N_\rho)_\alpha^\beta$, and if $N \neq 0$ fix an orthonormal basis of intertwiners $T_e = T^i \in (\rho_\alpha\rho|\rho_\beta)$:

$$T^{i*}T^j = \delta_{ij}, \quad i, j = 1, \ldots, N, \tag{2.10}$$

(i.e. here and from now on the collective label e consists apart from its charge, source, and range also of a multiplicity index $i = 1, \ldots, N$. We shall never display the multiplicity indices, and adopt an implicit summation convention for i whenever T_e and T_e^* occur in the same formula). Then

$$\sum_e T_e T_e^* = \mathbf{1}, \tag{2.11}$$

where the summation extends over $r(e)$, while $s(e), c(e)$ are kept fixed. If ρ_α or $\rho = id$, choose $T_e = \mathbf{1}$. If $\rho_\beta = id$ (hence $\rho_\alpha = \bar{\rho}$), call $T_e =: R_\rho$.
The reduced space bundle is the sum of Hilbert spaces

$$\mathcal{H} = \bigoplus_{\rho \in \nabla_0} \mathcal{H}_\rho \tag{2.12}$$

equipped with the scalar product $\langle(\rho_1, \Psi_1), (\rho_2, \Psi_2)\rangle = \delta_{\rho_1\rho_2}\langle\Psi_1, \Psi_2\rangle$ induced from the scalar product of $\mathcal{H}_0$.
The reduced field bundle is the sum of vector spaces (extending over all superselection channels e of ∇_0)

$$\mathcal{F} = \bigoplus_e \ (e, \mathcal{A}) \tag{2.13}$$

with operators $(e, A) \in \mathcal{F}$ acting on states $(\rho_\alpha, \Psi) \in \mathcal{H}$ by (cf. (2.8))

$$(e, A)\ (\rho_\alpha, \Psi) = \delta_{\rho_\alpha s(e)}(r(e), T_e^* \rho_\alpha(A)\Psi). \tag{2.14}$$

Proposition [8, 9]: $\mathcal{F}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$. More specifically we have:
(i) $\| (e, A) \| \leq \| A \|$.
(ii) $\mathcal{A}$ is contained in $\mathcal{F}$ by the identification

$$A = \sum_{e, c(e)=id} (e, A). \tag{2.15}$$

(iii) The following two definitions for (e, A) to be localized in $\mathcal{O}$ (or: $(e, A) \in \mathcal{F}(\mathcal{O})$) are equivalent:
(1) (e, A) commutes with $\mathcal{A}(\mathcal{O})$ acting on $\mathcal{H}$.
(2) There are $\hat{\rho}$ equivalent to $\rho = c(e)$, $\hat{\rho}$ localized in $\mathcal{O}$, and $U \in (\hat{\rho}|\rho)$ unitary, such that $UA \in \mathcal{A}(\mathcal{O})$.
(iv) The product of $(e_i, A_i) \in \mathcal{F}$ with charges $\rho_i = c(e_i)$ is again in $\mathcal{F}$:

$$(e_2, A_2)\ (e_1, A_1) = \delta_{s(e_2)r(e_1)} \sum_{e,f} \ D_{e_1\circ e_2; f, e}\ \ (e, A_f) \in \mathcal{F}, \tag{2.16}$$

where the finite sum extends over all f with $s(f) = \rho_1, c(f) = \rho_2$ and all e with $s(e) = s(e_1), r(e) = r(e_2)$ and $c(e) = r(f)$ (that is, (e, A_f) do the same interpolation as the product on the left-hand-side, and carry charges contained in $\rho_1\rho_2$). The notation $e_1 \circ e_2$ indicates the condition $s(e_2) = r(e_1)$. In (2.16), with $\rho_\alpha = s(e_1)$:

$$A_f := T_f^* \rho_1(A_2) A_1 \in \mathcal{A}, \tag{2.17}$$

$$D_{e_1\circ e_2; f, e} := T_{e_2}^* T_{e_1}^* \rho_\alpha(T_f) T_e \in (r(e)|r(e)) = \mathbb{C}. \tag{2.18}$$

(v) The following commutation relations (exchange algebra) hold. Let $(e_i, A_i) \in \mathcal{F}(\mathcal{O}_i)$ have charges $c(e_i) = \rho_i$, and $r(e_1) = s(e_2)$. Then

$$(e_2, A_2)(e_1, A_1) = \sum_{e_2' \circ e_1'} R^{(\pm)}_{e_1\circ e_2; e_2' \circ e_1'} (e_1', A_1)(e_2', A_2) \ \text{ if} \left\{ \begin{array}{l} \mathcal{O}_2 > \mathcal{O}_1 \\ \mathcal{O}_1 > \mathcal{O}_2 \end{array} \right. \tag{2.19}$$

where $c(e'_i) = \rho_i$, $r(e'_2) = s(e'_1)$, $r(e'_1) = r(e_2)$, $s(e'_2) = s(e_1) \equiv \rho_\alpha$, and

$$R^{(\pm)}_{e_1\circ e_2;e'_2\circ e'_1} := T^*_{e_2} T^*_{e_1} \rho_\alpha \begin{pmatrix} \varepsilon(\rho_2,\rho_1) \\ \varepsilon(\rho_1,\rho_2)^* \end{pmatrix} T_{e'_2} T_{e'_1} \in (r(e_2)|r(e_2)) = \mathbb{C}. \tag{2.20}$$

(vi) If $(e,A) \in \mathcal{F}(\mathcal{O})$, then $(e,A)^* \in \mathcal{F}(\mathcal{O})$:

$$(e,A)^* = \frac{d(\rho_\beta)d(\rho)}{d(\rho_\alpha)} \eta(e)\ (e^*, \bar\rho(A^*)R_\rho), \tag{2.21}$$

where for $e = (\rho_\alpha, \rho, \rho_\beta)$ we set $e^* = (\rho_\beta, \bar\rho, \rho_\alpha)$, and

$$\eta(e) := \rho_\beta(R^*_\rho) T_{e^*} T_e \in (\rho_\beta|\rho_\beta) = \mathbb{C}. \tag{2.22}$$

(vii) If $(e_i, A_i) \in \mathcal{F}(\mathcal{O}_i)$, $\mathcal{O}_1 < \ldots < \mathcal{O}_n$, and $s(e_{i+1}) = r(e_i)$, $s(e_1) = r(e_n) = id$, then

$$(e_n, A_n) \ldots (e_1, A_1) = \prod_i \vartheta(\hat e_i)\ (\hat e_1, A_1) \ldots (\hat e_n, A_n), \tag{2.23}$$

and a similar equation, with an additional factor $\prod \omega(\rho_i)$, holds for $\mathcal{O}_n < \ldots < \mathcal{O}_1$. Here, for $e = (\rho_\alpha, \rho, \rho_\beta)$, we set $\hat e = (\bar\rho_\beta, \rho, \bar\rho_\alpha)$ and

$$\vartheta(e) := R^*_{\rho_\alpha} T^*_{\hat e} \bar\rho_\beta(\varepsilon(\rho_\alpha, \rho) T_e) R_{\rho_\beta} \in (id|id) = \mathbb{C}. \tag{2.24}$$

Remark: The last formula is a generalized "Weak Locality Condition" for vacuum expectation values of operators in the reduced field bundle. Combining (vii) and (vi), allows to express the complex conjugate of such a vacuum expectation value as the vacuum expectation value of charge conjugate operators proportional to

$$(\bar e, \bar\rho(A^*)R_\rho), \qquad \bar e = \hat e^* = \widehat{e^*} = (\bar\rho_\alpha, \bar\rho, \bar\rho_\beta),$$

times coefficients η, ϑ, etc. differing by $\prod \omega(\rho_i)$ for the two orderings of the localizations as in (vii). This difference can be compensated by appropriate complex transformations $\mathcal{V}(\pm i\pi)$ of the covariance group taking x in $-x$, provided a Spin-Statistics theorem relates the spin quantum numbers to the statistics phases ω, and complex covariance transformations can be defined. These conditions being satisfied for the case of conformal theories on the light-cone, one obtains the

TPC Theorem [9]: There is an anti-unitary operator Θ

$$\Theta\ (e,A)\Omega \propto \mathcal{V}(\pm i\pi)(\bar e, \bar\rho(A^*)R_\rho)\Omega \quad \text{if} \quad (e,A) \in \mathcal{F}(\mathcal{O}),\ \mathcal{O} \gtrless 0 \tag{2.25}$$

taking charge, source, and range of operators in $\mathcal{F}$ into their conjugates, and taking $\mathcal{F}(\mathcal{O})$ into $\mathcal{F}(-\mathcal{O})$. The vacuum state is Θ-invariant.

For various practical calculations, such as those leading to the TPC theorem, it is important to note that the numerical coefficients are not independent. In particular one has (for $e = (\rho_\alpha, \rho, \rho_\beta)$)

$$\eta(e)\eta(e)^* = \frac{d(\rho_\alpha)}{d(\rho_\beta)d(\rho)}\mathbb{1}, \quad \eta(e^*) = \chi(\rho)\frac{d(\rho_\beta)}{d(\rho_\alpha)}\eta(e), \tag{2.26}$$

$$\vartheta(e)\vartheta(e)^* = \mathbb{1}, \quad \vartheta(\hat{e}) = \frac{\chi(\rho_\beta)}{\chi(\rho_\alpha)\omega(\rho)}\vartheta(e)^*, \tag{2.27}$$

$$\vartheta(e)^*\eta(e) = \frac{\chi(\rho_\alpha)}{\chi(\rho_\beta)\chi(\rho)}\vartheta(\bar{e})^*\eta(\bar{e}), \tag{2.28}$$

where

$$\chi(\rho) = \chi(\bar{\rho})^* := \omega(\rho)\ R_{\bar{\rho}}^*\varepsilon(\bar{\rho},\rho)R_\rho = d(\rho)\ \bar{\rho}(R_{\bar{\rho}}^*)R_\rho \tag{2.29}$$

are characteristic phases which take values $+1$ resp. -1 if ρ is a selfconjugate real resp. pseudoreal sector, and which may be put to 1 by independent choice of R_ρ, $R_{\bar{\rho}}$ if ρ is inequivalent to its conjugate.

The structure constants R (2.20) of the exchange algebra (known as "braid matrices" [17, 29] in conformal field theory) and D (2.18) of the operator product expansion (known as "duality matrices" [29]) satisfy the polynomial ("braid" and "pentagon") equations known from conformal field theory; these equations are in fact nothing but the intertwiner identities (2.3) and (2.4) evaluated on "path spaces" of intertwiners $T_{e_1} \dots T_{e_n}$. Actually, the numerical values depend on the reference morphisms ∇_0 and the intertwiners T_e chosen in the definition. While their transformation behaviour is manifest, their actual values are of limited relevance. The intrinsic quantities are, e.g., eigenvalues and the Markov traces associated to the statistics (see Sect. 4). Thus, if $E \in \rho_1\rho_2(\mathcal{A})'$ is a minimal projection on an irreducible subrepresentation of $\rho_1\rho_2$ equivalent to ρ, i.e. if $E = TT^*$ where $T \in (\rho_1\rho_2|\rho)$ is an isometry, then the following equations hold:

$$\varepsilon(\rho_2,\rho_1)\varepsilon(\rho_1,\rho_2)E = \frac{\omega(\rho)}{\omega(\rho_1)\omega(\rho_2)}E, \tag{2.30}$$

$$R_1^*\bar{\rho}_1(E)R_1 = \frac{d(\rho)}{d(\rho_1)d(\rho_2)}. \tag{2.31}$$

The R matrices possess various symmetries, e.g.

$$R^{(+)}_{e_1\circ e_2;f_2\circ f_1} = \frac{\omega(\rho_\alpha)\omega(\rho_\gamma)}{\omega(\rho_\beta)\omega(\rho_\delta)}R^{(-)}_{e_1\circ e_2;f_2\circ f_1}, \tag{2.32}$$

where $\rho_\alpha = s(e_1) = s(f_2)$, $\rho_\beta = r(e_1) = s(e_2)$, $\rho_\gamma = r(e_2) = r(f_1)$, $\rho_\delta = r(f_2) = s(f_1)$, and

$$R^{(\pm)}_{e_1\circ e_2;f_2\circ f_1} = \frac{\eta(f_1)}{\eta(e_1)}R^{(\mp)}_{e_2\circ f_1^*;e_1^*\circ f_2} = \frac{\eta(f_1)\eta(f_2)}{\eta(e_1)\eta(e_2)}R^{(\pm)}_{f_1^*\circ f_2^*;e_2^*\circ e_1^*} \tag{2.33}$$

$$= \frac{\vartheta(\hat{e}_1)\vartheta(\hat{e}_2)}{\vartheta(\hat{f}_1)\vartheta(\hat{f}_2)} R^{(\pm)}_{\hat{e}_2\circ\hat{e}_1;\hat{f}_1\circ\hat{f}_2} \tag{2.34}$$

$$= \frac{\eta(f_1)\eta(f_2)}{\eta(e_1)\eta(e_2)} \frac{\vartheta(\bar{f}_1)\vartheta(\bar{f}_2)}{\vartheta(\bar{e}_1)\vartheta(\bar{e}_2)} R^{(\pm)}_{\bar{f}_2\circ\bar{f}_1;\bar{e}_1\circ\bar{e}_2}, \tag{2.35}$$

where for simplicity $\frac{1}{\eta}$, $\frac{1}{\vartheta}$ stand for η^{-1}, ϑ^{-1} if $N(e) > 1$.

In fact, it can be shown that (2.23) and (2.32) for $\rho_\alpha = id$ are equivalent to the "pentagon identity" in the following sense. Given a set of fusion rules and a collection of R matrices compatible with these fusion rules and satisfying the braid equations as well as (2.32) for $\rho_\alpha = id$ ("on-vacuum monodromy") for some phases ω. Then matrices D satisfying the pentagon equations with R exist if and only if the R matrices of the inversion braids $(\sigma_1 \ldots \sigma_{n-1}) \ldots (\sigma_1\sigma_2)\sigma_1$ take the values specified by (2.23): $\prod \vartheta(\hat{e})\delta_{e'\hat{e}}$ for some coefficients ϑ. Actually it is sufficient that the latter holds for $n \leq 5$, the remaining identities being a consequence, as well as the remaining equations (2.32).

The statement may be put differently. Given a set of fusion rules and a collection of R matrices satisfying the braid equations, the on-vacuum monodromy and the weak-locality property (2.23). Then it is possible to define abstract unitary resp. isometric operators $\varepsilon(\rho_1, \rho_2)$ and T_e by their action on path spaces (in the sense of [30]) allowed by the fusion rules, and to define "parallel transports" ρ on these operators, such that $\varepsilon(\rho_1, \rho_2)$ intertwines from $\rho_1\rho_2$ to $\rho_2\rho_1$ and T_e from ρ_β to $\rho_\alpha\rho$ and (2.3), (2.4) are satisfied.

3 Automorphisms

In this section we present some formulae for the behaviour of statistics phases (and thus selection rules for fractional spins) under the composition of generic sectors with abelian sectors (automorphisms), and give criteria for the existence of representatives $\tau \in [\tau]$ satisfying $\tau^\nu = id$ if $[\tau^\nu] = [id]$. These results have been derived in [18]. We first recall some well-known results [3].

Lemma: (i) The following four definitions for $\tau \in \Delta_0$ irreducible to be an automorphism are equivalent:

(1) τ possesses an inverse $\tau^{-1} \in \Delta_0$.

(2) τ^2 is irreducible.

(3) ε_τ is a scalar (hence $\varepsilon_\tau = \lambda(\tau) = \omega(\tau)$).

(4) $d(\tau) = 1$.

(ii) For $\tau \in \Delta_0$ an automorphism and $\rho \in \Delta_0$ irreducible, $\rho\tau \simeq \tau\rho$ are again irreducible, $d(\rho\tau) = d(\tau\rho) = d(\rho)$, and

$$\varepsilon(\rho, \tau)\varepsilon(\tau, \rho) = \varepsilon(\tau, \rho)\varepsilon(\rho, \tau) = \frac{\omega(\tau\rho)}{\omega(\tau)\omega(\rho)} =: \Omega_\tau(\rho). \tag{3.1}$$

(iii) The equivalence classes of automorphisms in Δ_0 define an abelian group Γ_0 by class multiplication: $[\tau_1][\tau_2] = [\tau_1\tau_2]$, and $[\tau]^{-1} = [\tau^{-1}] = [\bar{\tau}]$.

Then the phases $\Omega_\tau(\rho)$ are multiplicative both with respect to τ and ρ:

Proposition: Let $\rho, \rho_i \in \Delta_0$ be irreducible, $\tau, \tau_i \in \Delta_0$ automorphisms. Then
(i)

$$\Omega_{\tau_1\tau_2}(\rho) = \Omega_{\tau_1}(\rho)\Omega_{\tau_2}(\rho). \tag{3.2}$$

(ii) If ρ is equivalent to a subrepresentation of $\rho_1\rho_2$, then

$$\Omega_\tau(\rho) = \Omega_\tau(\rho_1)\Omega_\tau(\rho_2). \tag{3.3}$$

For the statistics phases of automorphisms one finds

Corollary: Let $\tau \in \Delta_0$ be an automorphism.
(i)

$$\omega(\tau^m) = \omega(\tau)^{m^2}. \tag{3.4}$$

(ii) Suppose $[\tau^\nu] = [id]$. Then

$$\omega(\tau)^{\nu^2} = \omega(\tau)^{2\nu} = 1. \tag{3.5}$$

If ν is odd, or if for some odd μ there is a fixpoint equivalence class of $\tau^\mu : [\tau^\mu\rho] = [\rho]$, or if τ has permutation group statistics, then

$$\omega(\tau)^\nu = 1. \tag{3.6}$$

Longo [23] has introduced the spectrum of the restriction of the left-inverse to $(\rho^\nu|id) \subset (\rho^{\nu+1}|\rho)$

$$\phi : (\rho^\nu|id) \to (\rho^\nu|id)$$

as an intrinsic characterization of the generic sector ρ. For $\rho = \tau$ an automorphism, this map is just $\omega(\tau)^\nu = \pm 1$. The significance of this sign is given by the following

Proposition: (i) Let $\tau \in \Delta_0$ be an automorphism such that $[\tau^\nu] = [id]$. If and only if $\omega(\tau)^\nu = 1$, there is $\tilde{\tau} \in [\tau]$ satisfying $\tilde{\tau}^\nu = id$.
(ii) Let a subgroup $\Gamma = \bigotimes_i \mathbb{Z}_{\nu_i}$ ($\mathbb{Z}_0 \equiv \mathbb{Z}$) of Γ_0 be generated by τ_i with $[\tau_i^{\nu_i}] = [id]$. If and only if $\omega(\tau_i)^{\nu_i} = 1$, one may choose $\tilde{\tau}_i \in [\tau_i]$ generating a subgroup of Δ_0 isomorphic to Γ by individual multiplication.

Remarks: (1) In conformal field theories, automorphisms with the obstruction, i.e. $\omega(\tau)^\nu = -1$, are encountered. In $SU(2)$ WZW models [31] of level k, the self-conjugate automorphisms have scaling dimensions $\frac{k}{4}$, thus by the spin-statistics theorem $\omega(\tau)^2 = (-1)^k$. More generally, in $SU(N)$ WZW models at odd level k there are automorphisms of order $\nu = N$, which have $\omega(\tau)^\nu = -1$. In contrast, in all unitary

minimal models [13] and coset models [32] of $SU(N)$ as well as in the WZW models of even level, the fixpoint condition of the Corollary applies, hence $\omega(\tau)^\nu = 1$ for all automorphisms of order ν, and the obstruction is absent.
(2) It is possible to construct two-dimensional space-time fields with conventional commutation relations from exchange fields on the light-cone [18]. These fields carry charges on either of the two chiral factors which are conjugate to each other up to an additional abelian "excess charge". The latter must be an *unobstructed* automorphism.

4 Markov Traces

Let ρ be an irreducible sector in Δ_0, ϕ its unique left-inverse

$$\phi(A) = R^* \bar{\rho}(A) R. \tag{4.1}$$

Proposition: (i) The map

$$\varepsilon_\rho^{(\infty)} : \sigma_i \mapsto \rho^{i-1}(\varepsilon(\rho,\rho)) \tag{4.2}$$

defines a homomorphism of $B_\infty = \bigcup B_n$ (with the natural embedding $B_n \subset B_{n+1}$) into $M_\infty = \bigcup \rho^n(\mathcal{A})'$.
(ii) The map

$$\varphi := \lim_{N\to\infty} \phi^N : M_\infty \to \mathbb{C} \tag{4.3}$$

converges and defines a positive trace state in M_∞.
(iii) The map

$$tr_\rho := \varphi \circ \varepsilon_\rho^{(\infty)} : B_\infty \to \mathbb{C} \tag{4.4}$$

is a positive Markov trace with the property

$$tr_\rho(b_1 b_2) = tr_\rho(b_1) tr_\rho(b_2) \;\text{ if} \begin{cases} b_1 \text{ is a word in } \sigma_1, \ldots, \sigma_{n-1}, \\ b_2 \text{ is a word in } \sigma_n, \ldots, \sigma_m. \end{cases} \tag{4.5}$$

(iv) tr_ρ is independent of the choice of ρ in its equivalence class, and

$$tr_{\bar{\rho}} = (tr_\rho \circ I)^* \tag{4.6}$$

where I is the homomorphism of the braid group generated by $\sigma_i \mapsto \sigma_i^{-1}$.

The property (4.5) implies the usual Markov property (choosing $b_2 = \sigma_n$ and using $tr_\rho(b_i) = \phi(\varepsilon_\rho) = \lambda(\rho)$), but is much stronger; it seems not to have attracted much attention so far. The usual Markov property in general does not determine the trace state: even if the spectrum $\mu_1, \ldots, \mu_r$ of ε_ρ is known, the values of tr_ρ on infinitely many braids, e.g.

$$\sigma_1^{\nu_1} \ldots \sigma_n^{\nu_n}, \qquad 2 \leq \nu_i \leq r-2,$$

may be independently chosen. It might turn out that with the stronger property (4.5) the Markov trace is determined in terms of finitely many parameters [23]. This would be of great importance for the general classification problem.

The central point in the classification of statistics by their Markov traces is the fact that not all values of the parameters *admit* positive Markov traces. In the case of permutation group statistics, only $d \in \mathbb{N}$ (or $d = \infty$) are compatible with positivity; for $d \in \mathbb{N}$ the Markov trace is just the ordinary trace over tensor powers of $\mathbb{C}^d$ [3], the latter playing the role of a representation space of the compact symmetry group [7]. In fact, (i)–(iii) of the Proposition and the discussion below remain valid, if ρ is only assumed irreducible and transportable, and if ϕ is any of its left-inverses (which form a compact convex set; in this contribution we shall not discuss the general theory of left-inverses, see e.g. [3, 27]). At least in the following special case, which covers the case of permutation group statistics, the quantization of the admissible values of the statistics parameters and its implications on the superselection rules for powers of ρ *imply* the existence of a conjugate and in particular the uniqueness of the left-inverse.

Let us now discuss the special case of a sector ρ such that ε_ρ has two eigenvalues

$$(\varepsilon_\rho - \mu_1)(\varepsilon_\rho - \mu_2) = 0, \tag{4.7}$$

which is the case, e.g., if ρ has permutation group statistics, or if ρ^2 has only two inequivalent subrepresentations ρ_1, ρ_2. (In the latter case $\mu_i^2 = \omega(\rho_i)\omega(\rho)^{-2}$.) Then the linear hull of the image of B_∞ under $\varepsilon_\rho^{(\infty)}$ reduces to a Hecke algebra [10]. The positive Markov traces on Hecke algebras are quantized [8, 10], and one obtains the following results (for $\mu_1 \neq \mu_2$):

Proposition: Let $E_i^{(n)}$ be the spectral projectors on the simultaneous eigenspaces of $\varepsilon_\rho(\sigma_l)$, $l < n$, with eigenvalues μ_i. Then there are two integers $k_1, k_2 \geq 2$ (one of which may be infinite, in which case some of the statements below trivialize), such that $E_i^{(k_i+1)} = 0$ and $E_i^{(k_i)} \neq 0$. These integers are related to the spectrum of ε_ρ and to the statistics parameter by

$$\frac{\mu_1}{\mu_2} = -\exp\left[\pm\tfrac{2\pi i}{k_1+k_2}\right] =: -q, \tag{4.8}$$

$$\omega(\rho) = -\mu_1 \exp\left[\mp\pi i\tfrac{k_1+1}{k_1+k_2}\right] = -\mu_2 \exp\left[\pm\pi i\tfrac{k_2+1}{k_1+k_2}\right], \tag{4.9}$$

$$d(\rho) = [k_1]_q = [k_2]_q, \tag{4.10}$$

where $[k]_q = \dfrac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}}$.

Continuing the analysis one computes

$$\varphi(E_i^{(k_i)}) = d(\rho)^{-k_i}, \tag{4.11}$$

$$\varepsilon_\rho(C_l)E_i^{(k_i)} = \mu_i^{l(l-1)}E_i^{(k_i)}, \quad l \leq k_i, \tag{4.12}$$

where $C_l = (\sigma_1 \dots \sigma_{l-1})^l$ is the generator of the center of B_l.

On the other hand, consider the following decomposition of unity:

$$\mathbf{1} = \sum_{\underline{\omega}} E_{\underline{\omega}}, \qquad E_{\underline{\omega}} = \sum_{\xi,\, \underline{\omega}(\xi)=\underline{\omega}} T_\xi T_\xi^*, \tag{4.13}$$

where $\xi = e_1 \circ \dots \circ e_n$ is a path of length n of superselection channels with $c(e_i) = \rho$ and $s(e_1) = id$, $\underline{\omega}(\xi) = (\omega(r(e_1)), \dots, \omega(r(e_n)))$ is the trajectory of statistics phases along the path, and $T_\xi = T_{e_1} \dots T_{e_n} \in (\rho^n | r(e_n))$ is the corresponding path intertwiner. One has, generalizing (2.30) and (2.31) [8, 27]

$$\varphi(T_\xi T_\xi^*) = \frac{d(r(e_n))}{d(\rho)^n}, \tag{4.14}$$

$$\varepsilon_\rho(C_l)T_\xi = \frac{\omega(r(e_n))}{\omega(\rho)^n} T_\xi. \tag{4.15}$$

Now one can show that a path ξ of length k_i has the trajectory $\underline{\omega}(\xi) = \underline{\omega}_i = (\omega(\rho)^l \mu_i^{l(l-1)})_{l=1,\dots,k_i}$ if and only if $\varepsilon_\rho(\sigma_n)T_\xi = \mu_i T_\xi$ for all $n < k_i$. The "if" statement is obvious from (4.15), while the "only if" statement makes repeated use of the eigenvalue equation (4.7). Hence the projectors $E_{\underline{\omega}_i}$ are the maximal projectors satisfying

$$\varepsilon_\rho(\sigma_n)E = \mu_i E,$$

and thus coincide with $E_i^{(k_i)}$ by definition of the latter:

$$E_i^{(k_i)} = \sum_{\xi,\, \underline{\omega}(\xi)=\underline{\omega}_i} T_\xi T_\xi^*.$$

Then by (4.11) and (4.14) we conclude that there is (for every $i = 1, 2$) precisely one path ξ_i with $\underline{\omega}(\xi_i) = \underline{\omega}_i$, ending at a sector τ_i with $d(\tau_i) = 1$, hence τ_i is an automorphism.

Corollary: For $i = 1, 2$ there are unique paths $\xi_i = e_{i1} \circ \dots \circ e_{ik_i}$ such that

$$E_i^{(k_i)} = T_{\xi_i} T_{\xi_i}^*. \tag{4.16}$$

The sectors $\rho_{il} = r(e_{il})$, $l \leq k_i$, contained in ρ^l have statistics phases

$$\omega(\rho_{il}) = \omega(\rho)^l \mu_i^{l(l-1)}, \tag{4.17}$$

and ρ_{ik_i} contained in ρ^{k_i} is an automorphism τ_i. In particular, $\tau_i^{-1} \circ \rho_{i(k_i-1)}$ is conjugate to ρ.

Corollary: If either of the automorphisms τ_i, say τ_1, is the vacuum sector id, then (by virtue of (4.17) and (4.9))

$$\omega(\rho)^{k_i^2} = \exp\left[\mp\pi i \tfrac{k_1^3 - k_1}{k_1+k_2}\right]. \tag{4.18}$$

There are realizations [31, 32] in conformal field theory of these trajectories in WZW and coset models based on $SU(N)$ at level L, where $N = k_1$, $L = k_2$.

Let us now discuss the special case of a sector ρ such that ρ^2 has three inequivalent irreducible subrepresentations τ, ρ_1, ρ_2, where τ is an automorphism. This situation has first been treated by Longo [23] for the self-conjugate case $\tau = id$ [1]. Then ε_ρ has three eigenvalues, which we assume all different:

$$(\varepsilon_\rho - \mu_0)(\varepsilon_\rho - \mu_1)(\varepsilon_\rho - \mu_2) = 0, \tag{4.19}$$

where $\mu_0^2 = \omega_\tau/\omega^2$, $\mu_k^2 = \omega_k/\omega^2$ by (2.30) with $\omega, \omega_\tau, \omega_k$ the statistics phases of ρ, τ, ρ_k respectively, $k = 1, 2$. Using (3.2) and (3.3) one can derive $\Omega_\tau(\rho) = \omega_\tau$. If we denote

$$G_i = \rho^{i-1}(\varepsilon_\rho) = (G_i^{-1})^*, \tag{4.20}$$

$$E_i = \rho^{i-1}(TT^*) = E_i^* = E_i^2, \tag{4.21}$$

where $T \in (\rho^2|\tau)$ is an isometry and hence E_i the projector onto the eigenvalue μ_0 of G_i, we find from the properties of statistics operators, intertwiners, and automorphisms the equations

$$\begin{gathered} E_i = \frac{\mu_0}{(\mu_0 - \mu_1)(\mu_0 - \mu_2)}(G_i - (\mu_1 + \mu_2) + \mu_1\mu_2\ G_i^{-1}), \\ E_i G_i = \mu_0 E_i, \end{gathered} \tag{4.22}$$

$$\begin{gathered} G_i G_{i\pm1} G_i = G_{i\pm1} G_i G_{i\pm1}, \\ E_i G_{i\pm1} G_i = \omega_\tau\ E_i G_{i\pm1}^{-1} G_i^{-1} = d\omega\mu_0\ E_i E_{i\pm1}, \end{gathered} \tag{4.23}$$

$$\begin{gathered} E_i G_{i\pm1} E_i = \omega\ d^{-1} E_i, \\ E_i E_{i\pm1} E_i = d^{-2} E_i, \end{gathered} \tag{4.24}$$

$$\left.\begin{aligned} G_i G_j = G_j G_i \\ E_i E_j = E_j E_i \end{aligned}\right\} \text{ if } |i - j| \geq 2, \tag{4.25}$$

where d is the statistical dimension of ρ. Multiplying (4.22) with E_{i+1} from both sides, and multiplying

$$G_i^{\pm1} E_{i+1} G_i^{\mp1} = G_{i+1}^{\mp1} E_i G_{i+1}^{\pm1}$$

(which holds for G, G^{-1}, and 1 in the place of E, and hence holds also for E) with E_i from the right and with $E_i^{(k)}$, the spectral projectors of G_i onto the eigenvalues

[1] The assumption of self-conjugacy will also be dropped in the published version of [23].

μ_k, $k = 1, 2$, from the left, one obtains identities among the parameters, which after some algebra read:

$$(\mu_1\mu_2)^2 = \omega_\tau, \tag{4.26}$$

$$d = \zeta^{-2}\mu_0\omega + \frac{\zeta^{-1}\omega - \zeta\omega^{-1}}{\zeta^{-1}\mu_1 - \zeta\mu_1^{-1}} \equiv \eta\ (1 + \frac{r - r^{-1}}{q - q^{-1}}), \tag{4.27}$$

where ζ is a fourth root of ω_τ chosen such that $\mu_1\mu_2 = -\zeta^2$, implying that $\eta := \zeta^{-2}\mu_0\omega$ is just a sign to be determined by positivity of d, and where we have introduced $q := \zeta^{-1}\mu_1$, $r := \zeta\mu_0^{-1}$. Re-normalizing

$$g_i = \zeta^{-1}G_i, \tag{4.28}$$

$$e_i = \eta d\ E_i, \tag{4.29}$$

the equations (4.22–27) turn into the defining relations of the Birman-Wenzl algebra C_∞ (as given in [24] with the two parameters q, r introduced above). The map $\phi : C_{n+1} \to C_n$ defined by $e_n x e_n = \eta d\ \phi(x) e_n$ coincides with the left-inverse in the identification via $\varepsilon_\rho^{(\infty)}$, and converges to a positive Markov trace on the Birman-Wenzl algebra.

Wenzl [24] gives a complete list of values q, r consistent with positivity of the Markov trace. Except for a continuous one-parameter solution $r = q$ (hence $d = 2$), which appears to be realized in conformal models with a $\mathbb{Z}_2$-orbifold construction from $U(1)$-current algebra, all other solutions are discrete with q being (up to a sign) simple roots of unity, and r some integer power thereof. There are self-conjugate ($\tau = id$) realizations of these series in conformal WZW models [31] with current algebras of orthogonal and symplectic Lie groups.

5 Statistics Characters

Consider a finite group G. Let R_i denote the inequivalent irreducible representations of G, and C_j the conjugacy classes of G. Then the character table

$$X_{ij} = \chi_i(g_j), \tag{5.1}$$

where χ_i is the character of R_i, and g_j a group element in C_j, satisfies

$$X_{0j} = 1, \quad X_{i0} = \dim(R_i) = d_i, \tag{5.2}$$

$$X_{ij}X_{kj} = \sum_m N_{ik}^m X_{mj}, \tag{5.3}$$

where R_0 is the trivial representation, C_0 the trivial conjugacy class, and N_{ik}^m are the multiplicities of R_m in $R_i \otimes R_k$. In particular

$$\sum_m N_{ik}^m d_m = d_i d_k. \tag{5.4}$$

The matrix $Y_{ij} = X_{ij}c_j$, $c_j = |C_j|^{1/2}$ satisfies

$$Y_{\bar{\imath}j} = Y_{i\bar{\jmath}} = Y^*_{ij}, \tag{5.5}$$

$$YY^\dagger = Y^\dagger Y = |G|\mathbb{1}, \tag{5.6}$$

$$N^m_{ik} = \sum_j \frac{Y_{ij}Y_{kj}Y^{-1}_{jm}}{Y_{0j}} = \frac{1}{|G|}\sum_j \frac{Y_{ij}Y_{kj}Y^*_{mj}}{Y_{0j}}, \tag{5.7}$$

where $R_{\bar{\imath}}$ is the representation conjugate to R_i, $C_{\bar{\jmath}}$ the inverse conjugacy class of C_j, and $|G| = \sum d_i^2 = \sum c_j^2$ the order of the group.

We shall now define a generalization of the group characters χ_i describing the superselection structure of a theory with braid group statistics.

Let $[\rho_i]$ denote the equivalence classes of irreducible superselection sectors in Δ_0, or some subset $\Delta \subset \Delta_0$ closed under conjugation and composition with subsequent reduction. Let $[\rho_{\bar{\imath}}] = [\bar{\rho}_i]$, $[\rho_0] = [id]$, and N^k_{ij} denote the multiplicity of $[\rho_k]$ in $[\rho_i\rho_j]$, $d_i = d_{\bar{\imath}} = d(\rho_i)$, $\omega_i = \omega_{\bar{\imath}} = \omega(\rho_i)$. The following is well known:

Lemma: (i)

$$N^k_{0j} = \delta_{jk}, \qquad N^k_{ij} = N^k_{ji} = N^{\bar{\jmath}}_{i\bar{k}} = N^{\bar{k}}_{\bar{\imath}\bar{\jmath}}. \tag{5.8}$$

(ii)

$$\sum_k N^k_{ij}N^m_{lk} = \sum_r N^r_{il}N^m_{rj}, \tag{5.9}$$

(iii)

$$\sum_k N^k_{ij}d_k = d_id_j. \tag{5.10}$$

In particular, for fixed i, j there are only finitely many k such that $N^k_{ij} \neq 0$.

Definition: Let $\rho_i \in [\rho_i]$, and ϕ_i the unique left-inverses of ρ_i. The matrices Y_{ij} are independent of the choice of $\rho_i \in [\rho_i]$:

$$Y_{ij} := d_id_j\ \phi_j(\varepsilon(\rho_j, \rho_i)^*\varepsilon(\rho_i, \rho_j)^*). \tag{5.11}$$

Lemma: (i)

$$Y_{0i} = Y_{i0} = d_i. \tag{5.12}$$

(ii)

$$Y_{ij} = Y_{ji} = Y^*_{i\bar{\jmath}} = Y_{\bar{\imath}\bar{\jmath}}. \tag{5.13}$$

(iii)

$$Y_{ij} = \sum_k N^k_{ij}\frac{\omega_i\omega_j}{\omega_k}d_k. \tag{5.14}$$

(iv)

$$\frac{1}{d_j}Y_{ij}Y_{kj} = \sum_m N^m_{ik}Y_{mj}. \tag{5.15}$$

Proof: (i) is obvious from the definition. (iii) follows by

$$\phi_j(\varepsilon(\rho_j,\rho_i)^*\varepsilon(\rho_i,\rho_j)^*\sum_e T_eT_e^*) = \sum_e \frac{\omega_i\omega_j}{\omega_k}\phi_j(T_eT_e^*) = \sum_k \frac{\omega_i\omega_j}{\omega_k}N_{ij}^k\frac{d_k}{d_id_j},$$

where $e=(\rho_j,\rho_i,\rho_k)$. The first equation of (ii) follows from (iii), while for the second we use (2.4) (with $\rho_3 = id$) to compute

$$\begin{aligned} Y_{ij} = \phi_i(Y_{ij}) &= d_id_j\ \phi_i[R_j^*\bar\rho_j(\varepsilon(\rho_j,\rho_i)^*\varepsilon(\rho_i,\rho_j)^*)R_j] \\ &= d_id_j\ R_i^*\bar\rho_i[\rho_i(R_j^*)\varepsilon(\bar\rho_j,\rho_i)\varepsilon(\rho_i,\bar\rho_j)\rho_i(R_j)]R_i \\ &= d_id_j\ R_j^*[\phi_i(\varepsilon(\bar\rho_j,\rho_i)\varepsilon(\rho_i,\bar\rho_j)]R_j = R_j^*Y_{\bar j i}^*R_j = Y_{i\bar j}^*. \end{aligned}$$

For (iv) one uses properties of left-inverses [3, 8, 27] to compute

$$\begin{aligned} \frac{1}{d_j}Y_{ij}Y_{kj} &= d_jd_id_k\ \phi_i(\varepsilon(\rho_i,\rho_j)^*\phi_k(\varepsilon(\rho_k,\rho_j)^*\varepsilon(\rho_j,\rho_k)^*)\varepsilon(\rho_j,\rho_i)^*) \\ &= d_jd_id_k\ \phi_i\phi_k(\varepsilon(\rho_k\rho_i,\rho_j)^*\rho_j(\textstyle\sum_e T_eT_e^*)\varepsilon(\rho_j,\rho_k\rho_i)^*) \\ &= d_jd_id_k \textstyle\sum_e \phi_i\phi_k(T_e\varepsilon(\rho_m,\rho_j)^*\varepsilon(\rho_j,\rho_m)^*T_e^*) \\ &= d_jd_id_k \textstyle\sum_e \phi_i\phi_k(T_eT_e^*)\phi_m(\varepsilon(\rho_m,\rho_j)^*\varepsilon(\rho_j,\rho_m)^*) \\ &= d_jd_id_k\sum_m N_{ik}^m\frac{d_m}{d_id_k}Y_{jm}\frac{1}{d_jd_m} = \sum_m N_{ik}^m Y_{mj}, \end{aligned}$$

where $e=(\rho_k,\rho_i,\rho_m)$.

We call the vectors Y_j with components Y_{ij} the "weight vectors" and χ_i with components $X_{ij} = \frac{1}{d_j}Y_{ij}$ the "statistics characters" of the sector $[\rho_i]$, because the latter satisfy the equations (5.2) and (5.3) like group characters, with the statistical dimensions $d(\rho_i)$ in place of the dimensions of representations $dim(R_i)$. Note that (iv) states that Y_j are simultaneous eigenvectors of the fusion matrices N_i (with matrix elements N_{ik}^m) with eigenvalues $X_{ij} = \frac{1}{d_j}Y_{ij}$.

The following statements are only meaningful if the number of inequivalent irreducible sectors is finite $N<\infty$.

Lemma: Any two weight vectors Y_l, Y_j are either orthogonal (in the natural metric $\langle\ ,\ \rangle$ of $\mathbb{C}^N$) or parallel:

$$\langle Y_l, Y_j\rangle = 0 \quad \text{or} \quad d_jY_l = d_lY_j. \tag{5.16}$$

Proof: Contracting (5.15) with Y_{lk}^* and using (5.8) and (5.13), one gets

$$d_j^{-1}Y_{ij}\langle Y_l, Y_j\rangle = \langle Y_l, N_iY_j\rangle = d_l^{-1}Y_{il}\langle Y_l,Y_j\rangle,$$

from which the claim follows.

We call sectors with weight vector Y_i parallel to Y_0 "degenerate". Thus for degenerate sectors

$$Y_{ij} = d_id_j. \tag{5.17}$$

We shall first discuss the case when there are no degenerate sectors except $[\rho_0] = id$, and then the general case.

Proposition: The following statements are equivalent:
(i) $[\rho_0] = id$ is the only degenerate sector.
(ii) The matrix Y is invertible.
(iii) The number $\sigma := \sum_i d_i^2 \omega_i^{-1}$ satisfies $|\sigma|^2 = \sum_i d_i^2$. Defining matrices

$$S := |\sigma|^{-1} Y, \quad T := \left(\frac{\sigma}{|\sigma|}\right)^{1/3} Diag(\omega_i) \tag{5.18}$$

these satisfy the algebra

$$SS^\dagger = TT^\dagger = \mathbb{1}_N, \tag{5.19}$$

$$TSTST = S, \tag{5.20}$$

$$S^2 = C, \quad TC = CT = T, \tag{5.21}$$

where $C_{ij} = \delta_{i\bar{j}}$ is the conjugation matrix. Moreover

$$N_{ik}^m = \frac{1}{|\sigma|^2} \sum_m \frac{Y_{ij} Y_{kj} Y_{mj}^*}{Y_{0j}} = \sum_m \frac{S_{ij} S_{kj} S_{mj}^*}{S_{0j}}. \tag{5.22}$$

Proof: (iii) implies (ii) implies (i) is obvious. Assume that (i) holds. We shall prove (iii). Contracting (5.15) with d_j we get

$$\langle Y_{\bar{i}}, Y_k \rangle = \sum_m N_{ik}^m \sum_j Y_{mj} d_j = N_{ik}^0 \sum_j d_j^2 = \delta_{\bar{i}k} \sum_j d_j^2,$$

hence $YY^\dagger = (\sum d_j^2)\mathbb{1}$. Next, contracting (5.14) with $d_j \omega_j^{-1}$ we get (with (5.8), (5.10))

$$Y_{ij} \frac{d_j}{\omega_j} = \sum_k \left(\sum_j N_{i\bar{k}}^{\bar{j}} d_j \right) \frac{\omega_i}{\omega_k} d_k = \sigma \cdot d_i \omega_i,$$

hence $|\sigma|^2 = \sum d_j^2$, proving (5.19). Finally, contracting (5.15) with $d_j \omega_j^{-1}$ we get

$$\sum_j Y_{ij} \frac{1}{\omega_j} Y_{kj} = \sigma \sum_m N_{ik}^m \omega_m d_m = \sigma \cdot \omega_i \omega_k Y_{ik}^*,$$

which after division by $\omega_i \omega_k$ and complex conjugation implies (5.20); (5.21) is easily derived from (5.13), and (5.22) from (5.15).

Remarks: (1) These equations generalize the properties of the character table (5.5), (5.6), and (5.7). However, for a nonabelian finite group, Y_{ij} is not symmetric, and a diagonal matrix T with the above properties does not exist. Thus, a theory without degenerate sectors yields a"self-dual" object that is more symmetric than a group. While in the case of a group, c_j and d_i are related by a generalized Fourier transformation, in the self-dual case at hand the "conjugacy classes" C_j are in $1:1$ correspondence with the "representations" R_i, and

$$c_j \equiv \text{“}|C_j|^{1/2}\text{”} = d_j \equiv \text{“}\dim(R_j)\text{”}.$$

(2) Conformal models [31, 32] with modular invariant partition functions [25] are of the self-dual type described in the Proposition. In these models, the matrices S and T given by (5.14) and (5.18) describe the modular transformations $\tau \mapsto -\tau^{-1}$, $\tau \mapsto \tau + 1$ of the Virasoro characters $\chi_i^{\rm Vir}(\tau) = q^{-\frac{c}{24}} Tr_i \; q^{L_0}$, $q = \exp 2\pi i\tau$, and $\sigma^*/|\sigma| = \exp 2\pi i \frac{c}{8}$. The surprising observation that the modular S matrix "diagonalizes the fusion rules", i.e. (5.15) and (5.22), was in this context first made by Verlinde [26] and proven in [29]. It is even more surprising that we could derive the same algebra on completely general grounds, only assuming the absence of degenerate sectors. The question arises:

Question: What is the physical significance of the algebra $(5.19-21)$ in a general low-dimensional quantum field theory, in the absence of conformal covariance and modular invariance?

We do not have an answer, but we expect it to touch upon some very deep physical duality concept, see e.g. [21].

Let us now turn to the case that there are degenerate sectors.

Lemma: A sector $[\rho_i]$ is degenerate if and only if the monodromy operator $\varepsilon(\rho_i, \rho_j)\varepsilon(\rho_j, \rho_i) = 1$ for every other sector $[\rho_j]$.

Corollary: (i) If $[\rho_i]$ is degenerate, then ρ_i has permutation group statistics, and $\omega_i = \pm 1$, $d_i \in I\!\!N$.
(ii) In high dimensions, where statistics are permutation group statistics, every sector is degenerate.

Proof: The "if" statement of the Lemma is obvious from the definition (5.11); for the "only if" statement remark by comparing (5.10) and (5.14) that $Y_{ij} = d_i d_j$ is only possible if

$$\frac{\omega_k}{\omega_i \omega_j} = 1 \quad \text{ whenever } \; N_{ij}^k \neq 0.$$

But $\omega_k/\omega_i\omega_j$ exhaust the spectrum of the monodromy operator. (i) of the Corollary follows from $\varepsilon_\rho = \varepsilon_\rho^*$ and the quantization of statistics parameters for permutation group statistics [3]; (ii) is obvious.

Remark: Although we have motivated our analysis by the example of the character table of a group, our definition (5.11) is "blind" for the symmetry group associated with permutation group statistics [7]. The latter must be identified "by hand", e.g. from its representation theory given by N_{ij}^k, while it would be most desirable to compute its character table directly from the statistics, by some formula as powerful in the presence of degenerate sectors as (5.11) is in their absence. In fact, it is well known [33] that every (finite) set of commuting fusion matrices N_i satisfying (5.8) and (5.9) (an abelian "hypergroup") possesses a system of simultaneous eigenvectors with the dual role of components and eigenvalues as in (5.15). However, in general no formula like (5.14) is known to determine these eigenvectors, and in particular it is impossible to assign phases ω_i to the elements of the hypergroup with similar properties as described here.

Lemma: Two sectors $[\rho_j]$, $[\rho_k]$ have parallel weight vectors $d_j Y_k = d_k Y_j$ if and only if there is some degenerate sector $[\rho_i]$ with $N_{ji}^k \neq 0$.

Proof: The claim becomes evident, if the first equality in the proof of the Proposition above is rewritten (replacing $i \to \bar{\jmath}$, $m \to i$):

$$\langle Y_j, Y_k \rangle = \sum_i N_{ji}^k \langle Y_0, Y_i \rangle.$$

Corollary: (i) The degenerate sectors are a subset of sectors with permutation group statistics, closed under composition with subsequent reduction, and conjugation.
(ii) Every "family" of sectors with parallel weight vectors is closed and irreducible under the composition with degenerate sectors with subsequent reduction.

The structure of the families is not yet completely understood. However, the following is suggested by our results [34]:

Conjecture: From [7] we know that the subset of degenerate superselection sectors, having permutation group statistics, allows the construction of a (Bose-Fermi $\mathbb{Z}_2$-graded) field algebra $\mathcal{F}$, such that $\mathcal{A} \subset \mathcal{F}$ is the fixpoint subalgebra of $\mathcal{F}$ with respect to the action of a compact global gauge group G, and the degenerate sectors of $\mathcal{A}$ are in $1:1$ correspondence with the irreducible representations of G realized in $\mathcal{F}$. Then the families of non-degenerate superselection sectors of $\mathcal{A}$ arise as irreducible sectors (in a sense to be generalized for graded local nets) of $\mathcal{F}$, and the weight vectors of this superselection structure are of the self-dual type described in the Proposition.

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