Nuclear Physics B312 (1989) 715–750
North-Holland, Amsterdam

EINSTEIN CAUSALITY AND ARTIN BRAIDS

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Received 13 May 1988
Revised 25 July 1988)

Within the restricted context of conformal $QFTh_2$ we present a systematic analysis of the exchange algebras of light-cone fields which result from the previously studied global decomposition theory of Einstein-causal fields. Although certain aspects of the representation theory of exchange algebras with their Artin braid structure-constants appear in our illustrative examples (minimal and WZW models), our main interests are algebraic aspects. We view the present work as a new non-lagrangian (non-hamiltonian) approach to non-perturbative QFTh.

1. Introduction

The subject of this paper is an attempt to overcome the longstanding problem of dynamics in the algebraic approach [1] to non-perturbative local Quantum Field Theory (QFTh). The basic structure of this approach is the "local net", i.e. the association of a C^* algebra of local observables with every space-time region. Two operators localized in regions with space-like separation are required to commute with each other. This is the axiom of locality (Einstein Causality). Space-time and inner symmetries are formulated algebraically in terms of endomorphisms of the local net. The representation theory led to the understanding of important concepts in local QFTh, such as superselection rules and the relation between spin and statistics. The theory of (massive) particle scattering has been given a rigorous meaning via the LSZ formalism [2]. Representations with temperature $T > 0$ equilibrium states have been established [3], which are inequivalent to the vacuum representation.

The most unsatisfactory feature of this algebraic approach, however, is its failure in specifying and describing interactions. While Einstein Causality states the *absence* of interactions at space-like distances, the dynamics of a theory should be encoded in the deviation from commutativity of operators localized with non-space-like separation.

This is precisely the idea of this paper. Taking advantage of the peculiar light-cone structure of conformal QFTh in two dimensions ($CQFTh_2$), we discuss

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(North-Holland Physics Publishing Division)

the "exchange algebra" as a non-commutative bilinear algebra of light-cone operators, which *entails* the property of Einstein Causality for local fields (provided some numerical relations are satisfied). At the same time it codifies true dynamical information in the sense that it specifies time-like non-commutativity in terms of monodromy behaviour of n-point functions. Referring directly to conformal fields at the critical point, it is of genuinely non-perturbative character. Moreover, it contains explicit information about the "fusion rules" and implicit information about the anomalous scaling dimensions of the fields.

It is our intention to conceptualize earlier work of the authors [4] who were originally led to exchange algebras from the global decomposition theory of local operators, which in turn resulted – as a seasonable homologue of today's "conformal blocks" – from the resolution of the "Einstein Causality Paradox" [5]. There is a second path to exchange algebras pursued by Fröhlich [6], who started from the euclidean formulation and studied the monodromy properties of multivalued euclidean Green functions. We acknowledge many stimulating discussions with J. Fröhlich – who also introduced us into some of the mathematics, in particular Jones' work – about the questions which await an answer through the study of these algebraic structures, and new questions arising.

We put particular emphasis on the consistency constraints on conformal-exchange algebras, and thus on local algebras to be constructed from them. There are two types of consistency conditions, which are non-trivial to satisfy on the one hand, and sufficiently restrictive on the other hand, to single out only "a few" out of "all imaginable dynamics" of CQFTh_2. These include important classes of previously classified models. The consistency conditions are: (i) the associativity of the exchange operator algebra, which requires the "structure constants" to define representations of the Artin braid groups; and (ii) the compatibility with conformal transformation behaviour, which imposes additional relations on the braid-representation matrices.

We shall not try to tackle the complete classification problem of solutions to these conditions. We shall rather exploit the virtues of the rich structure implied by the conditions. In particular, we can give operator product expansions a meaning off-vacuum which is as satisfactory as on-vacuum, thus filling a gap in the old "bootstrap program".

2. Concepts and notions of CQFTh_2

Let us define a conformal quantum field theory in two dimensions by the following postulates, to be considered as working hypotheses rather than axioms.

(1) The conformal group has a unitary implementation U in Hilbert space. In particular there are hermitian infinitesimal generators $P_{\pm}$, $D_{\pm}$, $K_{\pm}$ commuting with

a basis of field operators $\phi_\alpha(x) = \phi_{\alpha^+\alpha^-}(x_+, x_-)$ as follows

$$i[P_+, \phi_\alpha(x_+, \cdot)] = \partial_+ \phi_\alpha(x_+, \cdot),$$
$$i[D_+, \phi_\alpha(x_+, \cdot)] = (x_+ \partial_+ + d_+)\phi_\alpha(x_+, \cdot),$$
$$i[K_+, \phi_\alpha(x_+, \cdot)] = (x_+^2 \partial_+ + 2x_+ d_+)\phi_\alpha(x_+, \cdot), \tag{2.1}$$

and likewise for P_-, D_-, K_-. Here $x_\pm = x^0 \pm x^1$ are the light-cone coordinates, and $d_\pm$ are the light-cone scaling dimensions of the conformal field ϕ_α.

The light-cone momenta $P_\pm$ have nonnegative spectra. There is a unique vacuum state Ω annihilated by $P_\pm$, $D_\pm$, $K_\pm$.

(2) The stress–energy tensor is a local field $\Theta_{\mu\nu}(x) = \Theta_{\nu\mu}(x)$ which is conserved: $\partial^\mu \Theta_{\mu\nu} = 0$, which has canonical dimension 2, and the time components of which are the energy–momentum densities: $\int dx^1\, \Theta_{0\nu}(x) = P_\nu$.

The following are, by now, standard consequences.

(i) [7] $\Theta_{\mu\nu}$ is traceless, and can be split into two light-cone fields $\Theta_+(x_+)$, $\Theta_-(x_-)$ of light-cone dimensions $d_+ = 2$, $d_- = 2$, respectively. Both are Lie fields, and their moments

$$L_n = \tfrac{1}{4}(-1)^n \int dx\,(x-i)^{1+n}(x+i)^{1-n}\Theta(x), \tag{2.2}$$

satisfy two independent Virasoro algebras (one for either light-cone; we omitted the label $+/-$). In particular

$$P = \tfrac{1}{2}\int dx\,\Theta(x) = L_0 + \tfrac{1}{2}(L_{-1} + L_1),$$

$$D = \tfrac{1}{2}\int x\,dx\,\Theta(x) = i(L_{-1} - L_1),$$

$$K = \tfrac{1}{2}\int x^2\,dx\,\Theta(x) = L_0 - \tfrac{1}{2}(L_{-1} + L_1). \tag{2.3}$$

The "compact picture" with its "radial quantization" (analyticity in "radial tubes" in Wightman's spirit) is obtained from this by a complex Möbius transformation $\zeta(x) = (x-i)/(1-ix)$, which maps the real axis onto the unit circle. This transformation is not a physical symmetry of the theory, but rather an isometric mapping from one theory into another. We make no use of the compact picture.

(ii) [8] Since the Weyl inversion I: $x \to -1/x$ is contained in the conformal group, $K = U(I)PU(I)^+$ has the same nonnegative spectrum as P. In particular the generators $L_0 = \frac{1}{2}(P+K)$ ("conformal Hamiltonians", one for either light-cone) of the compact subgroup U(1) of $SL(2,\mathbb{R})/Z_2$ have nonnegative spectrum.

(iii) [9] The conformal fields (2.1) organize into families $[\alpha^+\alpha^-]$ such that the orthogonal subspaces $\mathscr{H}_{[\alpha^+\alpha^-]} = \mathrm{span}\{\phi_{\alpha^+\alpha^-}(x)\Omega | \phi_{\alpha^+\alpha^-} \in [\alpha^+\alpha^-]\}$ are irreducible representation spaces of the Virasoro algebra $\mathrm{Vir}_+ \otimes \mathrm{Vir}_-$. For every representation, denoted also $[\alpha^+\alpha^-]$, there is a primary field of dimensions $h^+ = h(\alpha^+)$, $h^- = h(\alpha^-)$, while all other conformal fields belonging to $[\alpha^+\alpha^-]$ (quasiprimary fields) have dimensions $d^\pm = h^\pm + n^\pm$, $n^\pm \in \mathbb{N}$.

(iv) [5] The integration of the infinitesimal transformation behaviour (2.1) is in general impossible for a local field $\phi_{\alpha^+\alpha^-}$ as a whole. In fact there occur complex phases in the law for special conformal transformations $\mathsf{T}_b x = x/(1-bx)$ beyond the singular point, that depend on the states between which $\phi_{\alpha^+\alpha^-}$ is evaluated. Introducing orthogonal projectors $P_{\beta^+\beta^-}$ onto $\mathscr{H}_{[\beta^+\beta^-]}$ it is found from the phases of the three-point functions $(\phi_{\beta^+\beta^-}\Omega, \phi_{\alpha^+\alpha^-}\phi_{\gamma^+\gamma^-}\Omega)$ – which are in turn determined by the spectrum condition – that

$$U_\pm(a)\phi_{\alpha^+\alpha^-}(x_\pm, \cdot)U_\pm(a)^+ = \phi_{\alpha^+\alpha^-}(x_\pm + a, \cdot)$$

$$U_\pm(\lambda)\phi_{\alpha^+\alpha^-}(x_\pm, \cdot)U_\pm(\lambda)^+ = \lambda^{d_\alpha \pm}\phi_{\alpha^+\alpha^-}(\lambda x_\pm, \cdot)$$

$$U_\pm(b)P_{\beta^+\beta^-}\phi_{\alpha^+\alpha^-}(x_\pm, \cdot)P_{\gamma^+\gamma^-}U_\pm(b)^+$$

$$= \sigma^{\alpha^\pm}_{\beta^\pm\gamma^\pm}(b, x_\pm) \cdot P_{\beta^+\beta^-}\phi_{\alpha^+\alpha^-}(\mathsf{T}_b x_\pm, \cdot)P_{\gamma^+\gamma^-} \tag{2.4}$$

where

$$\sigma^\alpha_{\beta\gamma}(b, x) = (1 - b(x + i\varepsilon))^{-d_\alpha + h_\gamma - h_\beta}(1 - b(x - i\varepsilon))^{-d_\alpha - h_\gamma + h_\beta}$$

$$= |1 - bx|^{-2d_\alpha} \exp\left[2\pi i\, \mathrm{sig}(b)\theta(bx - 1)(h_\beta - h_\gamma)\right]; \tag{2.5}$$

θ is the step function, and $h_{\beta^\pm}$, $h_{\gamma^\pm}$ the primary dimensions of the representations $[\beta^+\beta^-]$, $[\gamma^+\gamma^-]$. The dot in eq. (2.4) stands for the other light-cone variable. In particular $\Theta_\pm$ and $U_\pm$ commute with the projectors $P_{\beta^+\beta^-}$. The center of the conformal group, generated by $Z^\pm = \exp 2\pi i L_0^\pm$, is represented nontrivially as

$$Z^\pm P_{\beta^+\beta^-} = \exp(2\pi i h_{\beta^\pm})P_{\beta^+\beta^-}$$

$$Z^\pm P_{\beta^+\beta^-}\phi_{\alpha^+\alpha^-}P_{\gamma^+\gamma^-}(Z^\pm)^+ = \exp\left[2\pi i(h_{\beta^\pm} - h_{\gamma^\pm})\right]P_{\beta^+\beta^-}\phi_{\alpha^+\alpha^-}P_{\gamma^+\gamma^-}, \tag{2.6}$$

thus the quantum symmetry group is the universal covering $\widetilde{\mathrm{SL}(2,\mathbb{R})}_+ \otimes \widetilde{\mathrm{SL}(2,\mathbb{R})}_-$ of the Möbius group.

(v) [9] Decompose a local n-point function $W = (\Omega, \phi_{\alpha_1^+\alpha_1^-} \ldots \phi_{\alpha_n^+\alpha_n^-}\Omega)$ into a sum of non-local functions

$$W_{\xi^+\xi^-} = \left(\Omega, \phi_{\alpha_1^+\alpha_1^-}P_{\beta_1^+\beta_1^-} \ldots P_{\beta_{n-1}^+\beta_{n-1}^-}\phi_{\alpha_n^+\alpha_n^-}\Omega\right), \tag{2.7}$$

ξ^+, ξ^- referring to the "channel" of successive projectors. Then $W_{\xi^+\xi^-} = F^+_{\xi^+} \cdot F^-_{\xi^-}$ factorizes into "conformal blocks" F_{ξ^+} depending on α^+, β^+, x_+ only, and F_{ξ^-} depending on α^-, β^-, x_- only.

This justifies [4] the factorization of the projected fields as operators

$$P_{\beta^+\beta^-}\phi_{\alpha^+\alpha^-}(x_+, x_-)P_{\gamma^+\gamma^-} = \left(P_{\beta^+}a^+_{\alpha^+}(x_+)P_{\gamma^+}\right) \otimes \left(P_{\beta^-}a^-_{\alpha^-}(x_-)P_{\gamma^-}\right),$$

$$\mathscr{H}_{[\beta^+\beta^-]} = V_{[\beta^+]} \otimes V_{[\beta^-]},$$

$$P_{\beta^\pm}a^\pm_{\alpha^\pm}P_{\gamma^\pm} \equiv \left(a^\pm_{\alpha^\pm}\right)_{\beta^\pm\gamma^\pm} : V_{[\gamma^\pm]} \to V_{[\beta^\pm]}. \tag{2.8}$$

We have thus obtained the field $\phi_{\alpha^+\alpha^-}$ as a bilinear combination of light-cone fields $(a^+_{\alpha^+})_{\beta^+\gamma^+}$ and $(a^-_{\alpha^-})_{\beta^-\gamma^-}$. It is an immediate question what locality, i.e. space-like commutativity of 2-dimensional fields $\phi_{\alpha^+\alpha^-}$, means in terms of light-cone fields $(a_\alpha)_{\beta\gamma}$. The latter cannot just commute with each other, formally since they carry their projectors with them, physically since then time-like commutativity were also implied. In sect. 3 we shall discuss as a most natural structure (see also ref. [6]) a bilinear "exchange algebra" satisfied by two collections of light-cone fields. It will have space-like commutativity of local fields as a consequence, while realizing non-trivial dynamics at time-like distances.

For this purpose we have to make one further structural assumption about the CQFTh_2 aimed at. Let us state it as follows. The non-vanishing 3-point functions $(\phi_{\beta^+\beta^-}\Omega, \phi_{\alpha^+\alpha^-}\phi_{\gamma^+\gamma^-}\Omega)$ define a set of formal, associative and commutative, "fusion rules"

$$[\alpha^+\alpha^-][\gamma^+\gamma^-] = \bigoplus_{\beta^+} \bigoplus_{\beta^-} [\beta^+\beta^-], \tag{2.9}$$

which specify the channels contributing to a local n-point function. Our assumption is

(3) for every $[\alpha^+\alpha^-], [\gamma^+\gamma^-]$ the sum in eq. (2.9) is finite.

The assumption is satisfied, e.g. in the minimal models ($c < 1$) [9], the Thirring model ($c = 1$) [10], the Wess–Zumino–Witten (WZW) models ($c \geq 1$) [11], and in a number of further classified models, characterized by various additional symmetries ($N = 1, 2$ supersymmetry [11], Z_N-symmetry [12], ...). It seems, however, not to be a feature of general CQFTh_2; instead, it is essentially this extremely untypical property of finiteness going along with a maximal domain of analyticity of the Wightman functions, which distinguishes the various classified "oases" scattered in the vast "desert" of $c \geq 1$ theories, generally inaccessible due to infinite fusion rules. Though, *formally*, the assumption (3) can easily be relaxed, the analytic properties are expected to change drastically.

Note that, in the above, $[\alpha^+\alpha^-]$ are not necessarily inequivalent Virasoro representations. Different fields corresponding to the same representation are counted separately in the fusion rules. There may be a larger symmetry (Kac–Moody- or supersymmetry) collecting families of $[\alpha^+\alpha^-]$ (with common primary dimensions mod $\mathbb{Z}$ in the Kac–Moody case, mod $\frac{1}{2}\mathbb{Z}$ in the supersymmetric case) into one irreducible representation of the enlarged symmetry.

3. Exchange field theory

The preceding discussion has provided the motivation for the following, model-independent definition of a conformal "exchange field theory". Here the term "exchange" makes the distinction from local field theory.

We have in mind a family of primary and quasiprimary conformal light-cone fields a_α, belonging to irreducible representations $[\alpha]$ in $V_{[\alpha]}$ of the Virasoro algebra, and a set of associative and commutative (formal) fusion rules

$$[\alpha][\gamma] = \bigoplus_{\beta}^{\text{finite}} [\beta]. \tag{3.1}$$

In particular, there are a vacuum sector V_0, a vacuum state $\Omega \in V_0$, and a vacuum representation [0] such that $[\alpha][0] = [\alpha]$.

For every γ, and for every β contributing to this sum, there is an intertwining light-cone field, depending on a one-dimensional variable $x = x_\pm$

$$(a_\alpha)_{\beta\gamma}(x) \equiv P_\beta a_\alpha(x) P_\gamma \colon V_\gamma \to V_\beta . \tag{3.2}$$

If in agreement with these fusion rules an operator product $P_{\beta_0} a_{\alpha_1}(x_1) P_{\beta_1} \dots P_{\beta_{n-1}} a_{\alpha_n}(x_n) P_{\beta_n}$ does not identically vanish, we shall call this product, and the corresponding multi-index $(\boldsymbol{\beta\alpha}) \equiv (\beta_0, \alpha_1, \dots, \alpha_n, \beta_n)$, "admissible". All admissible operator products are assumed linearly independent operator-valued functions of their light-cone variables. Multiplication in the field algebra is associative.

The fields $(a_\alpha)_{\beta\gamma}$ are supposed to behave under the respective groups of translations, dilatations, and special conformal transformations, like the projected local fields (2.4)

$$U(a) a_\alpha(x) U(a)^{-1} = a_\alpha(x+a),$$

$$U(\lambda) a_\alpha(x) U(\lambda)^{-1} = \lambda^{d_\alpha} a_\alpha(\lambda x),$$

$$U(b)(a_\alpha)_{\beta\gamma}(x) U(b)^{-1} = \sigma^\alpha_{\beta\gamma}(b,x)(a_\alpha)_{\beta\gamma}(\mathrm{T}_b x), \tag{3.3}$$

where the first two equations are irrespective of β and γ. The notations are as in eq. (2.4).

In particular

$$Z|_{V_\beta} = e(h_\beta),$$

$$Z(a_\alpha)_{\beta\gamma}(x)Z^+ = e(h_\beta - h_\gamma)(a_\alpha)_{\beta\gamma}(x), \tag{3.4}$$

where we have introduced the notation

$$e(h) = \exp(2\pi i h).$$

There is a field $\mathscr{T}(x)$ (which is just a rescaled version of $\Theta(x)$ such that $(\Omega, \mathscr{T}(x_1)\mathscr{T}(x_2)\Omega) = \frac{1}{2}c(x_1 - x_2 - i\varepsilon)^{-4}$) generating the conformal transformations like in eq. (2.3). The commutators of a_α with higher moments L_n, $|n| > 1$, eq. (2.2), take all fields belonging to $[\alpha]$ irreducibly into each other.

The usual notion of "locality" is meaningless in a light-cone theory. It is replaced by the following.

Exchange algebra. There are matrices $[R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}(x_1, x_2)]_{\beta_1\beta_1'}$ for $x_1 \neq x_2$ such that

$$P_{\beta_0}a_{\alpha_1}(x_1)P_{\beta_1}a_{\alpha_2}(x_2)P_{\beta_2}$$
$$= \sum_{\beta_1'} \left[R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}(x_1, x_2)\right]_{\beta_1\beta_1'} P_{\beta_0}a_{\alpha_2}(x_2)P_{\beta_1'}a_{\alpha_1}(x_1)P_{\beta_2}. \tag{3.5}$$

In particular, for $x_1 \neq x_2$

$$P_\beta a_\alpha(x_1)P_\gamma\mathscr{T}(x_2) = \mathscr{T}(x_2)P_\beta a_\alpha(x_1)P_\gamma, \tag{3.6}$$

and in case there is a Kac–Moody current $j(x)$

$$\tilde{P}_\beta a_\alpha(x_1)\tilde{P}_\gamma j(x_2) = j(x_2)\tilde{P}_\beta a_\alpha(x_1)\tilde{P}_\gamma,$$

where $\tilde{P}$ project onto the irreducible representation spaces of the enlarged symmetry.

Since all fields a_α, $\alpha \in [\alpha]$, are obtained from the corresponding primary field by appropriate short-distance limits of products with $\mathscr{T}(x)$, it follows that the matrices R depend on α_i only through $[\alpha_i]$. In the Kac–Moody case, if $[\alpha_i]$ and $[\alpha_i']$ belong to the same enlarged representation, the respective R matrices also coincide.

Proposition 1. The matrices $R^{(\beta_1\beta_2)}_{(\alpha_1\alpha_2)}(x_1, x_2)$ depend on the light-cone variables only through $s_{12} = \mathrm{sig}(x_1 - x_2)$. Calling

$$R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}(x_1 > x_2) =: R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}, \tag{3.7a}$$

one has

$$R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}(x_1 < x_2) = \left[R^{(\beta_0\beta_2)}_{(\alpha_2\alpha_1)}\right]^{-1}. \tag{3.7b}$$

Proposition 2. Define diagonal matrices

$$\left[\phi^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}\right]_{\beta\beta'} = \delta_{\beta\beta'} \exp 2\pi i\left(h_\beta - \tfrac{1}{2}\left(h_{\beta_0} + h_{\beta_2}\right)\right), \tag{3.8}$$

which depend on α_1, α_2 through β running over the values for which $(\beta_0\alpha_1\beta\alpha_2\beta_2)$ are admissible. Then

$$\phi^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)} R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)} \phi^{(\beta_0\beta_2)}_{(\alpha_2\alpha_1)} R^{(\beta_0\beta_2)}_{(\alpha_2\alpha_1)} = \mathbf{1}. \tag{3.9}$$

In particular "on the vacuum" ($\beta_2 = [0]$), where the size of the matrices is 1×1

$$R^{(\beta 0)}_{(\alpha_1\alpha_2)} R^{(\beta 0)}_{(\alpha_2\alpha_1)} = \exp 2\pi i\left(h_\beta - h_{\alpha_1} - h_{\alpha_2}\right). \tag{3.10}$$

Proposition 3. The exchange matrices $[R^{(\beta\beta')}_{(\alpha\alpha')}]_{\gamma\gamma'}$ satisfy

$$\sum_{\beta_1''}\left[R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}\right]_{\beta_1\beta_2''}\left[R^{(\beta_1''\beta_3)}_{(\alpha_1\alpha_3)}\right]_{\beta_2\beta_2'}\left[R^{(\beta_0\beta_2')}_{(\alpha_2\alpha_3)}\right]_{\beta_1''\beta_1'}$$
$$= \sum_{\beta_2''}\left[R^{(\beta_1\beta_3)}_{(\alpha_2\alpha_3)}\right]_{\beta_2\beta_2''}\left[R^{(\beta_0\beta_2'')}_{(\alpha_1\alpha_3)}\right]_{\beta_1\beta_1'}\left[R^{(\beta_1'\beta_3)}_{(\alpha_1\alpha_2)}\right]_{\beta_2''\beta_2'} \tag{3.11}$$

for all admissible multi-indices $(\beta_0\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3)$ and $(\beta_0\alpha_3\beta_1'\alpha_2\beta_2'\alpha_1\beta_3)$.

Before we prove these propositions, let us make some remarks.

The operator algebra (3.5) translates into analytic properties of Wightman n-point functions of light-cone fields. The latter are defined by complex conformal transformations as analytic functions in a large complex domain (the "conformal tube") of analyticity. There are cyclically ordered real points ("conformal Jost points") $(x_\nu > x_{\nu+1} > \cdots > x_{\nu-1}$, or $x_\nu < x_{\nu+1} < \cdots < x_{\nu-1})$ which lie inside this domain, while permuted real points lie on the boundary. The exchange algebra expresses n-point functions at permuted real points as linear combinations of n-point functions at the original points. This again enlarges the domain of analyticity into $\widetilde{\mathring{\mathbb{C}}^n}$, where $\circ$ denotes the omission of all points of coincidence $x_i = x_j$, $i \neq j$, and $\sim$ denotes the universal covering, since $\mathring{\mathbb{C}}^n$ is not simply connected. Actually, the theory makes use of a finite covering only. For these functions the matrices R describe the analytic exchange $x_i x_{i+1}$ with positive orientation of two neighboring variables, while $R_{(\alpha_1\alpha_2)}R_{(\alpha_2\alpha_1)}$ describe the monodromy $x_i \sim x_{i+1}$ (with positive orientation) around a branch point [4]. Vice versa, it can be shown from first principles of CQFTh_2 that the exchange algebra is a consequence of the domain of analyticity of conformal blocks being $\widetilde{\mathring{\mathbb{C}}^n}$ [6].

The identity of proposition 3 has obviously the structure of a Yang–Baxter equation [14] for a vertex or SOS type lattice model [15]: read α as a generalized "orientation" of a link, and β as the corresponding "heights" of the adjacent plaquettes. Solutions to the Yang–Baxter equations are well known, and the dependence on a rapidity or spectral parameter must be eliminated by appropriate limits [4]. Possibly there exist more solutions of eq. (3.11). Given a solution of eq. (3.11), one must check whether diagonal phase matrices ϕ exist satisfying eq. (3.10). If they do, most of the dimensional trajectories $h(\beta)$ of the corresponding model can be read off their entries. We shall give important examples in sect. 7.

Proof of Proposition 1. The first statement follows from the transformation laws under $U(a)$ and $U(\lambda)$ since, due to the latter, $R(x_1, x_2)$ must be translation- and scale-invariant quantities. The second statement becomes evident by solving eq. (3.5) for the operator products appearing on the r.h.s.

Proof of Proposition 2. The statement follows from the fact that transformations with $U(b)$ may change the sign $s_{12} = \mathrm{sig}(x_1 - x_2)$. In fact: $\mathsf{T}x_1 - \mathsf{T}x_2 = (x_1 - x_2)/(1 - bx_1)(1 - bx_2)$ implies $\mathsf{T}s_{12} = \mathrm{sig}(\mathsf{T}x_1 - \mathsf{T}x_2) = s_{12} \cdot \mathrm{sig}(1 - bx_1) \cdot \mathrm{sig}(1 - bx_2)$.

First applying a special conformal transformation $U(b)$ to eq. (3.5) yields an expression of $P_{\beta_0} a_{\alpha_1}(\mathsf{T}x_1) P_\beta a_{\alpha_2}(\mathsf{T}x_2) P_{\beta_2}$ in terms of $P_{\beta_0} a_{\alpha_2}(\mathsf{T}x_2) P_{\beta'} a_{\alpha_1}(\mathsf{T}x_1) P_{\beta_2}$. Re-expressing the latter via the exchange algebra in terms of $P_{\beta_0} a_{\alpha_1}(\mathsf{T}x_1) P_{\beta''} a_{\alpha_2}(\mathsf{T}x_2) P_{\beta_2}$, comparing coefficients and taking care of all phase factors stemming from the conformal transformation, yields the equation

$$\delta_{\beta\beta''} = \sum_{\beta'} e\left[\mathrm{sig}(b)\left(-h(\beta_0) + h(\beta) + h(\beta') - h(\beta_2)\right)\left(\theta(bx_1 - 1) - \theta(bx_2 - 1)\right)\right]$$
$$\times \left[R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}(s_{12})\right]_{\beta\beta'} \left[R^{(\beta_0\beta_2)}_{(\alpha_2\alpha_1)}(\mathsf{T}s_{21})\right]_{\beta'\beta''},$$

which in the case $\mathrm{sig}(1 - bx_1) = \mathrm{sig}(1 - bx_2) \Rightarrow \mathsf{T}s_{21} = -s_{12}$, $\mathrm{e}[\cdot] = 1$, is the identity eq. (3.7). In the case $\mathrm{sig}(1 - bx_1) = -\mathrm{sig}(1 - bx_2) \Rightarrow \mathsf{T}s_{21} = s_{12} =: s$, $(\theta(bx_1) - \theta(bx_2)) = \mathrm{sig}(b)s_{12}$, it is rewritten as

$$\mathbf{1} = \left[\phi^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}\right]^s R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}(s) \left[\phi^{(\beta_0\beta_2)}_{(\alpha_2\alpha_1)}\right]^s R^{(\beta_0\beta_2)}_{(\alpha_2\alpha_1)}(s).$$

In both cases, $s = +1$ or $s = -1$, this is equivalent to eq. (3.10).

Proof of Proposition 3. The statement expresses the associativity of the field algebra. It is sufficient to consider triple operator products $P_{\beta_0} a_{\alpha_1}(x_1) \times P_{\beta_1} a_{\alpha_2}(x_2) P_{\beta_2} a_{\alpha_3}(x_3) P_{\beta_3}$ at different points $x_i \neq x_j$, $i \neq j$. There are two ways to express these products in terms of $P_{\beta_0} a_{\alpha_3}(x_3) P_{\beta_1'} a_{\alpha_2}(x_2) P_{\beta_2'} a_{\alpha_1}(x_1) P_{\beta_3}$, corresponding to two inequivalent representations of the permutation (13) in terms of transpositions (12) and (23). This results in the conditions

$$\sum_{\beta_1''} \left[R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}(s_{12})\right]_{\beta_1\beta_1''} \left[R^{(\beta_1''\beta_3)}_{(\alpha_1\alpha_3)}(s_{13})\right]_{\beta_2\beta_2'} \left[R^{(\beta_0\beta_2')}_{(\alpha_2\alpha_3)}(s_{23})\right]_{\beta_1''\beta_1'}$$
$$= \sum_{\beta_2''} \left[R^{(\beta_1\beta_3)}_{(\alpha_2\alpha_3)}(s_{23})\right]_{\beta_2\beta_2''} \left[R^{(\beta_0\beta_2'')}_{(\alpha_1\alpha_3)}(s_{13})\right]_{\beta_1\beta_1'} \left[R^{(\beta_1'\beta_3)}_{(\alpha_1\alpha_2)}(s_{12})\right]_{\beta_2''\beta_2'}$$

for all admissible multiindices, and for all possible signs $s_{ij} = \mathrm{sig}(x_i - x_j)$. Using proposition 1 to express all matrices $R^{(\cdot)}_{(\cdot)}(+/-)$ in terms of $R^{(\cdot)}_{(\cdot)} = R^{(\cdot)}_{(\cdot)}(+)$, for every choice of signs the above equation can be rewritten, after appropriate matrix multiplication and relabelling of α_i, as eq. (3.11).

In order to avoid confusion we should indicate that we made some change of notation as compared with ref. [4]. First, we found it convenient to treat the two light-cones in like rather than opposite manner, and introduced $x_+ = u$, $x_- = -v$. (Then the euclidean section is $z_+ = -z_-^*$.) Second, the constant R matrices now describe positive rather than clockwise-oriented continuation. Third, the concept of a "scheme" has been replaced by the "fusion rules".

4. Local field theory

Let us now discuss the construction of a two-dimensional local field theory out of two light-cone exchange field theories. As suggested by sect. 2, we consider fields of the form

$$\phi_\alpha(x) := \sum_{\beta^+\gamma^+\beta^-\gamma^-} g^{(\alpha^+\alpha^-)}_{\beta^+\gamma^+;\beta^-\gamma^-} (a^+_{\alpha^+})_{\beta^+\gamma^+}(x_+) \otimes (a^-_{\alpha^-})_{\beta^-\gamma^-}(x_-) \tag{4.1}$$

with numerical coefficients $g^{(\cdot)}_{\cdot\cdot}$, which may take values different from 0, 1 since the light-cone fields a^+, a^- are so far defined without normalization. The point $x = (x^0, x^1)$ is defined by $x_\pm = x^0 \pm x^1$.

Proposition 4. Let the structure-constants matrices R^+, R^- referring to the exchange algebras of fields $(a^+_{\alpha^+})_{\beta^+\gamma^+}$, $(a^-_{\alpha^-})_{\beta^-\gamma^-}$ respectively satisfy the requirements of propositions 1–3, such that fields ϕ_α given by eq. (4.1) transform like conformal fields in two dimensions (eq. (2.4)). Then, two fields ϕ_{α_1} and ϕ_{α_2} satisfy the axiom of local (anti-)commutativity (Einstein causality)

$$\phi_{\alpha_1}(x_1)\phi_{\alpha_2}(x_2) = \varepsilon\phi_{\alpha_2}(x_2)\phi_{\alpha_1}(x_1)$$

$$\text{if} \quad (x_1 - x_2)^2 = (x_1 - x_2)_+ (x_1 - x_2)_- < 0 \tag{4.2}$$

($\varepsilon = +1$ or -1), if and only if

$$\sum_{\gamma^+} g^{(\alpha_1^+\alpha_1^-)}_{\beta_0^+\gamma^+;\beta_0^-\beta^-} g^{(\alpha_2^+\alpha_2^-)}_{\gamma^+\beta_2^+;\beta^-\beta_2^-} \left[R^{+(\beta_0^+\beta_2^+)}_{(\alpha_1^+\alpha_2^+)}\right]_{\gamma^+\beta^+}$$

$$= \varepsilon \sum_{\gamma^-} g^{(\alpha_2^+\alpha_2^-)}_{\beta_0^+\beta^+;\beta_0^-\gamma^-} g^{(\alpha_1^+\alpha_1^-)}_{\beta^+\beta_2^+;\gamma^-\beta_2^-} \left[R^{-(\beta_0^-\beta_2^-)}_{(\alpha_2^-\alpha_1^-)}\right]_{\gamma^-\beta^-} \tag{4.3}$$

for all $(\beta_0, \beta, \beta_2)^\pm$ such that $(\beta_1\alpha_1\beta\alpha_2\beta_2)^\pm$ are admissible.

Special cases

(1) Let the theory be parity invariant in the strong sense that $P(a_\alpha^+)_{\beta\gamma}(x)P = (a_\alpha^-)_{\beta\gamma}(x)$ (in particular the range of the labels α, β and the fusion rules coincide on the two light-cones, $\varepsilon = 1$, and $R^+ \equiv R^-$). Let $g^{(\alpha^+\alpha^-)}_{\beta^+\gamma^+;\beta^-\gamma^-} = \delta_{\alpha^+\alpha^-}\delta_{\beta^+\beta^-}\delta_{\gamma^+\gamma^-}\chi$, where $\chi = 1$, if $(\beta\alpha\gamma)^\pm$ are admissible, and $\chi = 0$ else. Then $P\phi(t, x)P = \phi(t, -x)$, and eq. (4.3) reduces to

$$\left[R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}\right]_{\beta^-\beta^+} = \left[R^{(\beta_0\beta_2)}_{(\alpha_2\alpha_1)}\right]_{\beta^+\beta^-}, \tag{4.4}$$

i.e. the R matrices for $[\alpha_2] \leftrightarrow [\alpha_1]$ are transpose to each other (which is consistent with propositions 2 and 3), and symmetric for $[\alpha_1] = [\alpha_2]$. The special solutions in subsect. 7.4 have this property.

(2) Let there be a 1:1 mapping i of $\{\alpha^+\}$ onto $\{\alpha^-\}$, and a 1:1 mapping j of $\{\beta^+\}$ onto $\{\beta^-\}$, compatible with the fusion rules, but not necessarily taking the right "vacuum label" into the left "vacuum label". Let $g^{(\alpha^+\alpha^-)}_{\beta^+\gamma^+;\beta^-\gamma^-} = \delta_{i(\alpha^+)\alpha^-}\delta_{j(\beta^+)\beta^-}\delta_{j(\gamma^+)\gamma^-}\chi$. Then eq. (4.3) reduces to

$$\left[R^{-(j(\beta_0)j(\beta_2))}_{(i(\alpha_2)i(\alpha_1))}\right]_{j(\beta)j(\beta')} = \varepsilon\left[R^{+(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}\right]_{\beta'\beta}. \tag{4.5}$$

If R^+ satisfy the requirements of propositions 1–3, then so do R^- defined through eq. (4.5). In particular, if the + and − exchange algebras coincide and the mappings i, j describe a symmetry of the fusion rules, then eq. (4.5) describes a symmetry of the R matrices. Conversely, R matrices with a symmetry of eq. (4.5) allow for the construction of "non-diagonal" local fields out of two coinciding exchange algebras. Examples will be given in subsect. 7.4 too.

Proof of proposition 4. The claim is immediately proved if eq. (4.1) is inserted into the l.h.s. of eq. (4.2), and the exchange algebra relations are applied. At space-like distances, the R^+ matrices and the R^- matrices occur with opposite exponents.

It is also evident that at time-like distances the R^+ and R^- matrices occur with the same exponent. Expressing R^- via eq. (4.3) by R^+, we see that essentially the squares of R^+, i.e. the monodromy matrices, describe the non-commutative behaviour at time-like distances. The examples show that the condition (4.3) can easily be satisfied if only the + exchange algebra is given, and that solutions less trivial than in case (1), which are of particular interest in the context of D-E type modular-invariant partition functions of minimal models, may be detected.

At this point, locality has been traded by virtue of proposition 4 for a set of simple numerical equations. We emphasize, however, that local CQFTh_2 has not become a trivial issue, since along with the exchange algebra came the conditions of proposition 1–3. Their restrictive and predictive power should not be underestimated. The rest of this paper will address these topics.

5. Representations of braid groups

The structure constants R of an exchange algebra define matrix representations of the braid groups B_n [16], which are the fundamental groups of $\mathring{\mathbb{C}}^n/\mathrm{S}_n$. The braid group B_n (see below) is generated by elements σ_i, $i=1,\dots,n-1$, (and their inverses), satisfying

$$\sigma_i\sigma_j=\sigma_j\sigma_i \quad \text{if} \quad |i-j|\geq 2,$$

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}. \tag{5.1}$$

There is a natural homomorphism π: $\mathrm{B}_n\to\mathrm{S}_n$, $\sigma_i\to\tau_i$, τ_i the transposition $(i,i+1)$, of the braid group into the symmetric group.

The representation matrices $\rho_i=\rho(\sigma_i)$ induced from an exchange algebra describe the transposition of two neighboring operators $a_{\alpha_i}(x_i)$, $a_{\alpha_{i+1}}(x_{i+1})$ within an admissible n-point operator product

$$P_{\beta_0}\alpha_{\alpha_1}(x_1)P_{\beta_1}\dots P_{\beta_{n-1}}a_{\alpha_n}(x_n)P_{\beta_n}.$$

These keep β_0, β_n fixed, while α_i $(i=1,\dots,n)$ are permuted and β_i $(i=1,\dots,n-1)$ are changed according to the fusion rules. Hence, the matrix indices of the representation matrices run over all admissible $(\boldsymbol{\beta\alpha})=(\beta_0\alpha_1\dots\alpha_n\beta_n)$ with β_0, β_n fixed, and $(\alpha_1,\dots,\alpha_n)$ permutations of a fixed n-tuple $A=(\alpha_1^0,\dots,\alpha_n^0)$.

The following proposition is a concise reformulation of propositions 2 and 3.

Proposition 5. Given an exchange algebra satisfying the requirements of propositions 1–3. Choose $A=(\alpha_1^0,\dots,\alpha_n^0)$ and β_0, β_n.

(1) Then the matrices

$$[\rho_i]_{(\boldsymbol{\beta\alpha})(\boldsymbol{\beta'\alpha'})}:=\delta_{\tau_i\boldsymbol{\alpha},\boldsymbol{\alpha'}}\delta_{\beta_1\beta_1'}\dots\widehat{\delta_{\beta_i\beta_i'}}\dots\delta_{\beta_{n-1}\beta_{n-1}'}\left[R^{(\beta_{i-1}\beta_{i+1})}_{(\alpha_i\alpha_{i+1})}\right]_{\beta_i\beta_i'}, \tag{5.2}$$

define a representation $\rho^{(A,\beta_0,\beta_n)}(\sigma_i):=\rho_i$ of B_n ($\widehat{\ }$ denotes omission of a term).

(2) Let

$$[\varphi_i]_{(\boldsymbol{\beta\alpha}),(\boldsymbol{\beta'\alpha'})}:=\delta_{\boldsymbol{\alpha},\boldsymbol{\alpha'}}\delta_{\boldsymbol{\beta},\boldsymbol{\beta'}}e\left(h_{\beta_i}-\tfrac{1}{2}\left(h_{\beta_{i-1}}+h_{\beta_{i+1}}\right)\right). \tag{5.3}$$

Then we have

$$\varphi_i\rho_i\varphi_i\rho_i=\mathbf{1}. \tag{5.4}$$

Example. Consider three fields a_{α_1}, a_{α_2}, a_{α_3}, and choose β_0 arbitrary, $[\beta_3]=[0]$. Denote $h(\alpha_i)=h_i$, $h(\beta_0)=h_0$.

For a given permutation (ijk) of (123) referring to an operator product $P_{\beta_0}a_{\alpha_i}a_{\alpha_j}a_{\alpha_k}P_{\beta_3}$ the intermediate projector on $[\beta_2]=[\alpha_k]$ is fixed while only $\beta_1=:\beta$ is a free label (ranging over values specified by the fusion rule $[\alpha_j][\alpha_k]=\oplus[\beta]$).

Thus, denote the matrix indices of ρ_i by $(ijk;\,\beta)$. Then

$$[\rho_2]_{(ijk;\,\beta)(ikj;\,\beta')} = R^{(\beta 0)}_{(\alpha_j\alpha_k)}\delta_{\beta\beta'},$$

$$[\rho_1]_{(ijk;\,\beta)(jik;\,\beta')} = \left[R^{(\beta_0\alpha_k)}_{(\alpha_i\alpha_j)}\right]_{\beta\beta'}, \tag{5.5}$$

$$[\varphi_1]_{(ijk;\,\beta)(ijk;\,\beta)} = e\big(h(\beta) - \tfrac{1}{2}(h_0 + h_k)\big),$$

$$[\varphi_2]_{(ijk;\,\beta)(ijk;\,\beta)} = e\big(h_k - \tfrac{1}{2}h(\beta)\big), \tag{5.6}$$

while all other matrix elements vanish. Now, from $\varphi_2\rho_2\varphi_2\rho_2 = \mathbf{1}$ we deduce

$$[\rho_2^2]_{(ijk;\,\beta)(ijk;\,\beta')} = R^{(\beta 0)}_{(\alpha_j\alpha_k)}R^{(\beta 0)}_{(\alpha_k\alpha_j)}\delta_{\beta\beta'} = e\big(h(\beta) - h_k - h_j\big)\delta_{\beta\beta'}$$

$$= e\big(\tfrac{1}{2}h_0 - h_j - \tfrac{1}{2}h_i\big)\cdot[\varphi_1]_{(ijk;\,\beta)(ijk;\,\beta')}.$$

Inserting φ_1 into $\varphi_1\rho_1\varphi_1\rho_1 = \mathbf{1}$ yields

$$\delta_{\beta\beta'} = \sum_{\beta''} e\big(-\tfrac{1}{2}h_0 + h_j + \tfrac{1}{2}h_k\big)[\rho_2^2]_{(ijk;\,\beta)(ijk;\,\beta)}[\rho_1]_{(ijk;\,\beta)(jik;\,\beta'')}$$

$$\times e\big(-\tfrac{1}{2}h_0 + h_i + \tfrac{1}{2}h_k\big)[\rho_2^2]_{(jik;\,\beta'')(jik;\,\beta'')}[\rho_1]_{(jik;\,\beta'')(jik;\,\beta')}$$

$$= e\big(-h_0 + h_i + h_j + h_k\big)[\rho_2^2\rho_1\rho_2^2\rho_1]_{\beta\beta'}, \tag{5.7}$$

and hence, using the braid relations,

$$(\rho_2\rho_1\rho_2)^2 = (\rho_1\rho_2)^3 = e(+h_0 - h_i - h_j - h_k)\mathbf{1}. \tag{5.8}$$

That this is a multiple of unity is no surprise, since $(\sigma_1\sigma_2)^3$ generates the center of B_3. Its precise value, however, is computed from eq. (5.8).

Now consider a field a_α with nonvanishing four-point functions, and the representation of B_4 belonging to $P_0a_\alpha a_\alpha a_\alpha a_\alpha P_0$. The matrices ρ_2 and ρ_3 are precisely what ρ_1 and ρ_2 were in the above, with a trivial permutation index and $h_0 = h_i = h_j = h_k = h_\alpha$. In particular $(\rho_3\rho_2\rho_3)^2 = e(-2h_\alpha)\mathbf{1}$. In this case we have, moreover, $\rho_1 = \rho_3$, so that $(\rho_1\rho_2\rho_3)^2 = e(-2h_\alpha)\mathbf{1}$. This is more than could be expected since the center of B_4 is generated by $(\sigma_1\sigma_2\sigma_3)^4$.

The example has shown that the phase conditions (3.9) may imply additional relations among the representation matrices ρ_i of the braid group alone, thus effectively representing some quotient B_n/Ideal. We believe that this is true in the general case, the resulting equations being comparable to Vafa's [17]. Less ambitiously, one may do without understanding of the group-theoretical significance, and just take the determinants of eq. (5.4). Since:

(i) in every representation of B_n all ρ_i, $i=1,\ldots,n-1$, have a common spectrum; and

(ii) for $[\beta_n]=[0]$ the matrix ρ_{n-1}^2 is a diagonal matrix with entries (3.10) given in terms of conformal dimensions, and multiplicities determined by the fusion rules, one immediately gets a system of linear equations

$$\sum c_\beta h_\beta = 0 \bmod \mathbb{Z}\left(c_\beta \in \mathbb{N}\right), \tag{5.9}$$

for the dimensional trajectory $h_\beta = h([\beta])$ which can be written down and solved without any knowledge about the precise form of the off-vacuum R matrices.

The advantage of the reformulation of exchange algebras in terms of braid-group representations as introduced in proposition 5 is that important field-theoretic manipulations are translated into mathematical standard manipulations with group representations.

Recall proposition 4, for example. The requirement (4.3) describes some projection of the representation $\rho_+ \otimes \rho_-^{-1}$ of B_n onto a one-dimensional subrepresentation (the trivial ($\varepsilon=1$) or the anti-symmetric ($\varepsilon=-1$)); here ρ^{-1} is the representation defined by $\rho^{-1}(\sigma_i)=(\rho(\sigma_i))^{-1}$. The failure of time-like (anti-)commutativity is reflected by the fact that the same projection cannot be expected to define a subrepresentation of $\rho_+ \otimes \rho_-$ at the same time.

So far, the precise forms of the representation matrices ρ (proposition 5) and the projection coefficients (proposition 4) refer explicitly to the sectors V_β of *conformal* QFTh. One may drop this reminiscence and imagine a generalization of the above possibly leading beyond CQFTh_2. Define an exchange algebra by a collection of braid-group representations acting on light-cone n-point functions. An analogue of eq. (5.4) is not required. Two-dimensional fields are then *defined* as one-dimensional subrepresentations ρ of $\rho_+ \otimes \rho_-^{-1}$, where exotic statistics (i.e. $\rho(\sigma_i^2)=\varepsilon^2 \neq 1$) may be admitted. This generalization would preserve the issue of explicit non-trivial time-like commutation behaviour of QFTh_2.

Let us, however, return to our original concept of *conformal* light-cone fields, and discuss further field-theoretic manipulations in the light of braid-group representations.

First there is the evident observation that the R matrices relevant for the tensor products of independent fields are just the tensor products of the R matrices of the respective factor fields, while the induced representations of the braid groups are just the tensor products of the representations induced by the factors. It is likely evident that multiplying R matrices by an overall (phase) factor χ preserves the representation conditions. Combining these two harmless operations, however, has nontrivial field-theoretic consequences. Referring to subsect. 7.3 for the discussion of this phenomenon, we want to just mention here that the factor χ influences the dimensional trajectories according to proposition 2. In particular the dimension of the "product field" will no longer equal the sum of the dimensions of the factor fields, as it should for an ordinary tensor product.

Second, there is a natural construction of new representations out of old, which exists only for braid groups. The idea is to combine several "threads" of the braid into one "strand", which (as we shall see in sect. 6) corresponds to the field-theoretic operation of short-distance operator products. For an explanation it is helpful to recall the pictorial description of the braid group (which actually stood at the historical origin of studying braids as topological objects in $\mathbb{R}^3$ [16]).

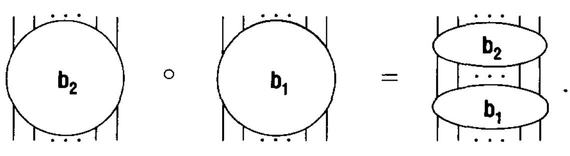
An n-braid b is described by a set of n non-intersecting curves $\boldsymbol{c}_j$: $[0,1] \to \mathbb{R}^3$, monotonous in the 2-component, with $\boldsymbol{c}_j(0) = (j,0,0)$, $\boldsymbol{c}_j(1) = (\pi_b(j),1,0)$, $\pi_b \in \mathrm{S}_n$, $j = 1,\dots,n$. Two such sets of curves describe the same braid, if they can be continuously deformed into each other without intersections. The natural description of the identity braid $\boldsymbol{e}$ is $\{\boldsymbol{c}_j(t) = (j,t,0)\}$, $\pi_e = e \in \mathrm{S}_n$. Typically

(5.10)

The composition law for $b_2 \circ b_1$ is given by functions

$$\tilde{c}[b_2 \circ b_1]_j(t) := \begin{cases} c[b_1]_j(2t), & t \le \frac{1}{2}, \\ c[b_2]_{\pi_{b_1}(j)}(2t-1) + e_2, & t \ge \frac{1}{2} \end{cases}$$

(hence $\tilde{c}[b_2 \circ b_1]_j(1) = ((\pi_{b_2} \circ \pi_{b_1})(j), 2, 0))$, the 2-components of which are rescaled by $\frac{1}{2}$ in order to obtain $c[b_2 \circ b_1]_j(t)$. Graphically this is just "linking b_2 on top of b_1"



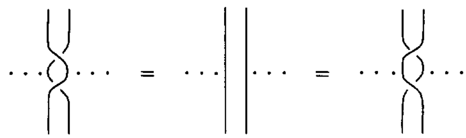
(5.11)

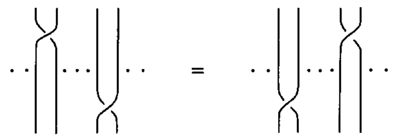
The projection π: $b \mapsto \pi_b$ is a group homomorphism of B_n in S_n.

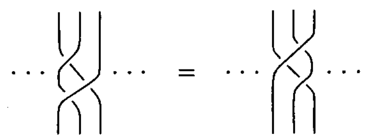
The generating braids σ_i (respectively σ_i^{-1}) are described by functions with a single intersection of the ith with the $(i+1)$th curve projected into the 1–2 plane, say at "time" t_0, such that $(e_3 \cdot c_i(t_0)) >$ [respectively $< (e_3 \cdot c_{i+1}(t_0))$]. Thus $\pi_{\sigma_i} = \tau_i =$ transposition of $(i, i+1)$

(5.12)

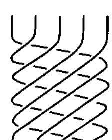
Every continuous deformation can be performed as a series of "Reidemeister moves", i.e. the following braid identities hold

$R_1: \sigma_i\sigma_i^{-1} = e = \sigma_i^{-1}\sigma_i$: 

$R_2: \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2$: 

$R_3: \sigma_i\sigma_{i+1}\sigma_1 = \sigma_{i+1}\sigma_i\sigma_{i+1}$:  (5.13)

The center of B_n is generated by $(\sigma_1 \ldots \sigma_{n-1})^n$

$(\sigma_1 \cdots \sigma_{n-1})^n =$  . (5.14)

The following propositions describe "strand products" of (representations of) B_n.

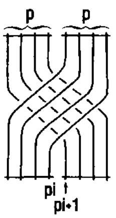
Proposition 6. There is a natural embedding I of B_ν in B_n, $n = p\cdot\nu$, which maps $\hat{\sigma}_i \in B_\nu$ onto

$$\Big(\prod_{(pi,0)}\sigma\Big)\Big(\prod_{(pi,1)}\sigma\Big)\cdots\Big(\prod_{(pi,i-1)}\sigma\Big)\cdots\Big(\prod_{(pi,1)}\sigma\Big)\Big(\prod_{(pi,0)}\sigma\Big)\in B_n, \qquad (5.15)$$

where

$$\Big(\prod_{(a,b)}\sigma\Big) := \prod_{k=0}^{b}\sigma_{a-b+2k},$$

are commutative products. Graphically

$$I(\hat{\sigma}_i) = \cdots$$


$\cdots$.

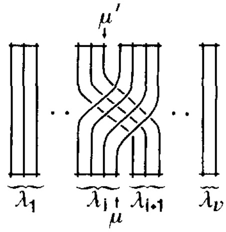
Proposition 7. Given a representation ρ of B_n in V. Let $\nu \le n$. Consider the set Λ of inequivalent partitions $(\lambda) = (\lambda_1,\ldots,\lambda_\nu)$, $\lambda_i \in \mathbb{N}$, of $n = \sum_{i=1}^{\nu}\lambda_i$, which are just permutations of some partition (λ^0) of n. Introduce an orthonormal basis $\{e_{(\lambda)}\}$ of $\mathbb{R}^N$, where $N = |\Lambda|$.

Let $\tau_i(\lambda) = (\lambda_1\ldots\lambda_{i+1}\lambda_i\ldots\lambda_\nu)$, $1\le i\le \nu-1$. The following defines a representation $\hat{\rho}$ of B_ν in $V\otimes\mathbb{R}^N$, depending on ρ and Λ.

For a pair $[i,(\lambda)]$ let $\mu = \sum_{j\le i}\lambda_j$, $\mu' = \sum_{j<i}\lambda_j + \lambda_{i+1}$, $\lambda = \min(\lambda_i,\lambda_{i+1}) - 1$. Set

$$\rho[i,(\lambda)] := \rho\bigg(\Big(\prod_{(\mu',0)}\sigma\Big)\cdots\Big(\prod_{(\mu',\lambda)}\sigma\Big)\cdots\Big(\prod_{(\mu,\lambda)}\sigma\Big)\cdots\Big(\prod_{(\mu,0)}\sigma\Big)\bigg), \qquad (5.16)$$

where the dots interpolate in unit steps from 0 to λ, from μ' to μ, and from λ to 0, respectively. Graphically

$$\rho[i,(\lambda)] = \rho\;\Bigg($$


$\Bigg)$.

Then $\hat{\rho}$ is given by

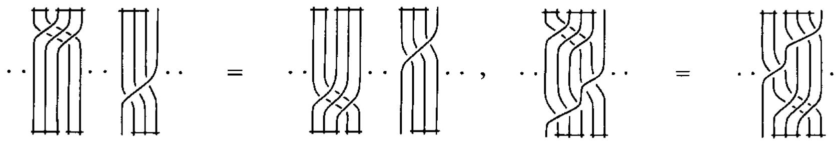
$$\hat{\rho}(\hat{\sigma}_i) := \sum_{(\lambda)}\rho[i,(\lambda)]\otimes e_{\tau_i(\lambda)}e^{\mathrm{T}}_{(\lambda)}. \qquad (5.17)$$

In the case $n = p \cdot \nu$, $(\lambda) = (p, \ldots, p)$, $N = 1$, the representation constructed in proposition 7 is nothing but the representation induced by the embedding of proposition 6.

Proof of propositions 6 and 7. All one has to show is that the objects $I(\hat{\sigma}_i)$, respectively $\hat{\rho}(\hat{\sigma}_i)$, commute for $|i-j| \geq 2$ and satisfy the defining braid relation R_3

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

These statements, however, are graphically evident from the following pictures of identities in B_n



(5.18)

6. Operator products in exchange field theory

An operator product

$$P_{\beta_0}\big(a_{\alpha_1}(y_1) P_{\beta_1} \ldots P_{\beta_{\nu-1}} a_{\alpha_\nu}(y_\nu)\big) P_{\beta_\nu}\big(a_{\alpha_{\nu+1}}(y_{\nu+1}) P_{\beta_{\nu+1}} \ldots P_{\beta_{n-1}} a_{\alpha_n}(y_n)\big) P_{\beta_n},$$

can be expressed in terms of

$$P_{\beta_0}\big(a_{\alpha_{\nu+1}}(y_{\nu+1}) P_{\beta_1'} \ldots P_{\beta'_{n-\nu-1}} a_{\alpha_n}(y_n)\big) P_{\beta'_{n-\nu}}\big(a_{\alpha_1}(y_1) P_{\beta'_{n-\nu+1}} \ldots P_{\beta'_{n-1}} a_{\alpha_\nu}(y_\nu)\big) P_{\beta_n},$$

by means of a product of $\nu(n-\nu)$ matrices ρ_i à la proposition 5. If $y_1, \ldots, y_\nu$ are all sufficiently close to x_1, and $y_{\nu+1}, \ldots, y_n$ are all sufficiently close to $x_2 \neq x_1$, then these matrices appear all with the *common* sign $s_{12} = \mathrm{sig}(x_1 - x_2)$. The matrix products so obtained are the R matrices for short-distance operator products of the type $P_{\beta_0}(a_{\alpha_1}(y_1) P_{\beta_1} \ldots P_{\beta_{\nu-1}} a_{\alpha_\nu}(y_\nu)) P_{\beta_\nu}$ with each other, and they automatically satisfy the appropriate braid relations ($\equiv$ associativity conditions). This is the field-theoretic analog of the "strand formation" and proposition 7. However, the resulting algebra is not a new exchange algebra in the precise sense of sect. 3: the operator products to be exchanged cannot readily be assigned to a particular (composite) field A_α, sandwiched among sector projectors. Instead, one must find linear combinations

$$P_{\beta_0} A_\alpha([x]) P_{\beta_\nu} = \sum_{\beta_1 \ldots \beta_{\nu-1}} \mathrm{C}^{(\beta_0;\, \alpha_1 \ldots \alpha_\nu;\, \beta_\nu)}_{\alpha;\, \beta_1 \ldots \beta_{\nu-1}} P_{\beta_0}\big(a_{\alpha_1}(y_1) P_{\beta_1} \ldots P_{\beta_{\nu-1}} a_{\alpha_\nu}(y_\nu)\big) P_{\beta_\nu}, \quad (6.1)$$

such that the new R matrices restricted to these combinations describe an algebra

$$P_{\beta_0}A_{\alpha_1}([x_1])P_{\beta_1}A_{\alpha_2}([x_2])P_{\beta_2}$$
$$= \sum_{\beta_1'}\left[R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)}(s_{12})\right]_{\beta_1\beta_1'}P_{\beta_0}A_{\alpha_2}([x_2])P_{\beta_1'}A_{\alpha_1}([x_1])P_{\beta_2}. \qquad (6.2)$$

Here, for the moment $[x]$ stands formally for the set $(y_1 \dots y_\nu)$ of arguments close to x.

It is well known [18] that operator products *on the vacuum* (i.e. $[\beta_\nu]=[0]$), after multiplication with appropriate scaling functions $f_\alpha(y,\partial_y)$, possess well defined short-distance limits. These behave precisely like light-cone field states $P_\alpha a_\alpha(x)\Omega$ under conformal transformations. The missing point in the early studies was that there was no way to define short-distance limits *off vacuum*.

Now, this gap is filled by the Exchange algebra. Its virtue is that it relates ν-point operator products $P_{\beta_0}a_{\alpha_1}(y_1)\dots a_{\alpha_\nu}(y_\nu)P_{\beta_\nu}$ applied to μ-point states $P_{\beta_\nu}a_{\alpha_{\nu+1}}(y_{\nu+1})\dots a_{\alpha_{\nu+\mu}}(y_{\nu+\mu})\Omega$ *linearly* to states $P_{\beta_0}a_{\alpha_{\nu+1}}(y_{\nu+1})\dots a_{\alpha_{\nu+\mu}}(y_{\nu+\mu})P_{\beta_\mu'}a_{\alpha_1}(y_1)\dots a_{\alpha_\nu}(y_\nu)\Omega$, i.e. the analytic behaviour of ν-point operator products can be inferred from the analytic behaviour of ν-point states, and short-distance limits that exist on-vacuum exist with the same scaling functions off-vacuum as well.

Note that in the short-distance operator products $P_{\beta_0}A_\alpha([x])P_{\beta_\nu}$ (eq. (6.1)) the label α refers to $[\alpha]$. The individual quasiprimary field $a_\alpha(x), \alpha\in[\alpha]$, is only selected by the choice of the appropriate scaling function $f_\alpha(y,\partial_y)$. For $\nu=2$, $f_\alpha(y_1,y_2,\partial_{y_1},\partial_{y_2})$ are given by [18]

$$N^{-1}Q^{n_\alpha}_{h_\alpha+d_{\alpha_1}-d_{\alpha_2},\,h_\alpha-d_{\alpha_1}+d_{\alpha_2}}(\partial_{y_1},\partial_{y_2})(y_1-y_2-i\varepsilon)^{d_{\alpha_1}+d_{\alpha_2}-h_\alpha}, \qquad (6.3)$$

where $Q^k_{n_1n_2}(z_1,z_2)=z_1^{1-n_1}z_2^{1-n_2}(\partial_{z_1}-\partial_{z_2})^k(z_1^{n_1+k-1}z_2^{n_2+k-1})$, $d_\alpha=h_\alpha+n_\alpha$, and N^{-1} is a normalization constant.

The problem in practice is the determination of the coefficients $\mathrm{C}^{(\beta_0;\,\alpha_1\dots\alpha_\nu;\,\beta_\nu)}_{\alpha;\,\beta_1\dots\beta_{\nu-1}}$, respectively the inverse expansion

$$P_{\beta_0}a_{\alpha_1}(y_1)P_{\beta_1}\dots P_{\beta_{\nu-1}}a_{\alpha_\nu}(y_\nu)P_{\beta_\nu}=\sum_{\alpha}^{\text{finite}}\mathrm{C}^{(\alpha,\beta)}_{\alpha}P_{\beta_0}A_\alpha([x])P_{\beta_\nu}. \qquad (6.4)$$

At least for $\nu=2$ these can be computed intrinsically, i.e. in terms of braid representation matrices (and scaling dimensions). This may illuminate the power of proposition 5. Consider the representation $\rho=\rho^{(A=(\alpha_1\alpha_2\alpha_3);\,\beta_0,0)}$of B_3 introduced in the example following proposition 5. Then

$$\left(P_{\beta_0}a_{\alpha_i}(y_1)P_{\beta_1}a_{\alpha_j}(y_2)P_{\alpha_k}\right)a_{\alpha_k}(x_2)\Omega$$
$$=\sum_{\beta_1'}\left[\rho_2^{s_{12}}\rho_1^{s_{12}}\right]_{(ijk;\,\beta_1)(kij;\,\beta_1')}P_{\beta_0}a_{\alpha_k}(x_2)\left(P_{\beta_1'}a_{\alpha_i}(y_1)P_{\alpha_j}a_{\alpha_j}(y_2)\right)\Omega. \quad (6.5)$$

The operator products in brackets on the r.h.s. can be identified with $P_{\beta_1'}A_{\beta_1'}([x_1])P_0$. The required coefficient matrix must "diagonalize" both $(\rho_2\rho_1)$ and $(\rho_2^{-1}\rho_1^{-1})$ simultaneously. From the example (eq. (5.8)) we know that $(\rho_1\rho_2\rho_1)\rho_2\rho_1 = e(H)\rho_2^{-1}$, and $(\rho_2\rho_1\rho_2)\rho_2^{-1}\rho_1^{-1} = \rho_2$, where $H = h_0 - h_i - h_j - h_k$. Thus comparing

$$\left(\sum_\beta e\left(-\tfrac{1}{2}H\right)[\rho_2\rho_1\rho_2]_{(kji;\,\alpha)(ijk;\,\beta)} P_{\beta_0} a_{\alpha_i}(y_1) P_\beta a_{\alpha_j}(y_2) P_{\alpha_k}\right) a_{\alpha_k}(x_2)\Omega$$

$$= \left[e\left(\tfrac{1}{2}H\right)\rho_2^{-1}\right]^{s_{12}}_{(kji;\,\alpha)(kij;\,\alpha)} P_{\beta_0} a_{\alpha_k}(x_2) P_\alpha A_\alpha([x_1])\Omega,$$

which follows from eq. (6.5), with the expected equation

$$\left(P_{\beta_0}A_\alpha([x_1])P_{\alpha_k}\right)a_{\alpha_k}(x_2)\Omega = R^{(\beta_0 0)}_{(\alpha\alpha_k)}(x_1, x_2) P_{\beta_0} a_{\alpha_k}(x_2) P_\alpha A_\alpha([x_2])\Omega,$$

one obtains

$$\mathrm{C}^{(\beta_0;\,\alpha_i\alpha_j;\,\alpha_k)}_{\alpha\beta} = e\left(-\tfrac{1}{2}H\right) R^{(\alpha 0)}_{(\alpha_j\alpha_i)}\left[R^{(\beta_0\alpha_j)}_{(\alpha_k\alpha_i)}\right]_{\alpha\beta} R^{(\beta 0)}_{(\alpha_k\alpha_j)},$$

$$R^{(\beta_0 0)}_{(\alpha\alpha_k)} = e\left(\tfrac{1}{2}H\right) R^{(\alpha 0)-1}_{(\alpha_i\alpha_j)},$$

$$R^{(\beta_0 0)}_{(\alpha_k\alpha)} = e\left(\tfrac{1}{2}H\right) R^{(\alpha 0)-1}_{(\alpha_j\alpha_i)}, \tag{6.6}$$

reproducing eq. (3.10). Note that both the braid relations (proposition 3) and the phase relations (proposition 2) have entered the argument essentially. Here we have computed the coefficients of an operator product $P_{\beta_0}a_{\alpha_i}P_\beta a_{\alpha_j}P_{\alpha_k}$ from the required algebra with the field $P_{\alpha_k}a_{\alpha_k}P_0$. The *same* operator product must satisfy an exchange algebra with all fields $P_{\alpha_k}a_{\alpha_l}P_\gamma$ simultaneously. This requirement needs further investigation.

As an example consider a theory of hermitian fields $(P_\beta a_\alpha P_\gamma)^+ = P_\gamma a_\alpha P_\beta$. Fix normalizations by the following equations

$$(\Omega, a_\alpha(x_1)a_{\alpha'}(x_2)\Omega) = \mathrm{sig}(\alpha)e\left(-\tfrac{1}{2}d_\alpha\right)\delta_{\alpha\alpha'}(x_1 - x_2 - i\varepsilon)^{-2d_\alpha},$$

$$\left(\Omega, a_{\alpha_1}(x_1)a_{\alpha_2}(x_2)a_{\alpha_3}(x_3)\Omega\right) = c_{\alpha_1\alpha_2\alpha_3}\prod_{i<j}(x_i - x_j - i\varepsilon)^{-d_i - d_j + d_k},$$

$$e\left(\tfrac{1}{4}\sum d_i\right)c_{\alpha_1\alpha_2\alpha_3} \in \mathbb{R}$$

$$c_{\alpha_1\alpha_3\alpha_2} = c_{\alpha_2\alpha_1\alpha_3} = (-)^{\Sigma(d_i - h_i)}c_{\alpha_1\alpha_2\alpha_3}$$

$$R^{(\alpha_1 0)}_{(\alpha_2\alpha_3)} = e\left(\tfrac{1}{2}(h_1 - h_2 - h_3)\right).$$

Unitary theories are characterized by all signs sig(α) of *all* quasiprimary fields being +1 [19].

Four-point functions have the well-known vacuum short-distance expansions

$$\left(\Omega, a_{\alpha_1}(x_1) a_{\alpha_2}(x_2) P_\beta a_{\alpha_3}(x_3) a_{\alpha_4}(x_4)\Omega\right)$$

$$= \left(\frac{x_2 - x_4}{x_1 - x_4}\right)^{d_1 - d_2} \left(\frac{x_1 - x_3}{x_1 - x_4}\right)^{d_4 - d_3} \Big/ (x_1 - x_2)^{d_1 + d_2} (x_3 - x_4 - i\varepsilon)^{d_3 + d_4}$$

$$\times \sum_{\beta \in [\beta]} e\left(\tfrac{1}{2} d_\beta\right)\left(c_{\alpha_1\alpha_2\beta} \operatorname{sig}(\beta) c_{\beta\alpha_3\alpha_4}\right)$$

$$\times x^{d_\beta} {}_2F_1(d_2 - d_1 + d_\beta, d_3 - d_4 + d_\beta; 2d_\beta; x), \tag{6.7}$$

which converge for $x_1 > x_2 > x_3 \approx x_4$; $x = (x_1 - x_2)(x_3 - x_4 - i\varepsilon)/(x_1 - x_3) \times (x_2 - x_4) \approx 0$.

The exchange algebra and eq. (6.6) imply the following off-vacuum short-distance expansions of the same functions

$$\sum_\beta \mathrm{C}_{\alpha\beta}^{(\alpha_1;\, \alpha_2\alpha_3;\, \alpha_4)} \left(\Omega, a_{\alpha_1}(x_1) a_{\alpha_2}(x_2) P_\beta a_{\alpha_3}(x_3) a_{\alpha_4}(x_4)\Omega\right)$$

$$= \left(\frac{x_2 - x_4}{x_1 - x_2}\right)^{d_1 - d_4} \left(\frac{x_1 - x_3}{x_1 - x_2}\right)^{d_2 - d_3} \Big/ (x_1 - x_4)^{d_1 + d_4} (x_2 - x_3 - i\varepsilon)^{d_2 + d_3}$$

$$\times \sum_{\alpha \in [\alpha]} e\left(\tfrac{1}{2} d_\alpha\right)\left(c_{\alpha_1\alpha\alpha_4} \operatorname{sig}(\alpha) c_{\alpha\alpha_2\alpha_3}\right)$$

$$\times (1 - x)^{d_\alpha} {}_2F_1(d_4 - d_1 + d_\alpha, d_3 - d_2 + d_\alpha; 2d_\alpha; 1 - x), \tag{6.8}$$

which converge for $x_1 > x_2 \approx x_3 > x_4$; $1 - x = (x_1 - x_4)(x_2 - x_3 - i\varepsilon)/(x_1 - x_3) \times (x_2 - x_4) \approx 0$.

It is crucial that the above expansions do not have the correct analytic properties term by term. Instead, the exchange algebra can only be satisfied by a most delicate interplay of the expansion coefficients, i.e. the quasiprimary three-point amplitudes and the signs $\operatorname{sig}(\alpha)$ signalling violation of unitarity of the field theory [19]. The evaluation of this interplay would constitute the final completion of the "conformal bootstrap program".

It is only fair to mention that ideas very similar to the "reduction" of strand-product representations – but in a different physical context – have been discussed earlier. Let us quote Karowski's approach to "boundstate S-matrices" (elaborated further by Kulish [20]), and the "fusion" of Yang–Baxter matrices of RSOS models [21]. The relevance of strand products was also observed by Fröhlich [6].

7. Examples

In this section we discuss a class of solutions to the conditions of propositions 1–3, which turn out to be the exchange algebras of the minimal models [9] and SU(2) WZW-models [11]. We develop the techniques discussed in general in the preceding sections in this specific context. In doing this, the power of our concepts will become apparent, both concerning the classification problem, and the qualitative and quantitative construction of conformal block functions. A counterexample will also be given, where a solution to the braid relations is incompatible with the conformal transformation behaviour, thus showing the relevance of both types of conditions.

7.1. THE SOLUTION

We give the exchange algebra for a single "elementary" field $a_\alpha \equiv a$ which interpolates according to the fusion rules

$$[\alpha][l] = [l-1] \oplus [l+1], \tag{7.1}$$

where l takes integer values $1, 2, \dots, q-1$. $[l-1] = [0]$ and $[l+1] = [q]$ are omitted from eq. (7.1). The sector $[l] = [1]$ may (not necessarily) be identified with the vacuum sector; in that case $[\alpha] = [\alpha][\text{vacuum}] = [2]$.

We found all solutions to eq. (3.11) compatible with the fusion rules (7.1) [4]

$$[R^{(11)}]_{22} = [R^{(q-1,q-1)}]_{q-2,q-2} =: \eta,$$

$$[R^{(l-1,l+1)}]_{ll} = [R^{(l+1,l-1)}]_{ll} =: \eta\omega,$$

$$[R^{(ll)}]_{l\mp1,l\mp1} = \eta(-\omega)^{1/2} \times \begin{pmatrix} -(-\omega)^{l/2} s(1)/s(l) & \lambda_l^{-1}\sqrt{s(l-1)s(l+1)/s^2(l)} \\ \lambda_l\sqrt{s(l-1)s(l+1)/s^2(l)} & (-\omega)^{-l/2} s(1)/s(l) \end{pmatrix}, \tag{7.2}$$

with

$$(-\omega) = \exp(2\pi i p/q), \qquad (p \text{ and } q \text{ coprime}),$$

$$s(l) := \sin(l\pi p/q). \tag{7.3}$$

The off-diagonal square roots in $R^{(ll)}$ are taken with the same sign, and $(-\omega)^{1/2} := \exp(2\pi i p/2q)$. Aiming at a parity-symmetric solution as in the special case (1) of proposition 4, we assume $\lambda_l = 1$. Then $R^{(ll)}$ are symmetric matrices, which are also unitary iff $p = \pm 1 \bmod q$, and η a phase. In fact, in a unitary theory all conformal blocks contributing to a local n-point function are – up to a common complex phase – real functions at the "Jost points", such that the inversely oriented exchange is described by the complex conjugate matrix; hence we should require $R^* = R^{-1}$, and R is symmetric $\Leftrightarrow$ R is unitary. Note, however, that unitarity of the field theory is a much deeper issue than just unitarity of the R matrices!

The solutions (7.2) possess the manifest symmetry $l \to q - l$

$$[R^{(q-l_0, q-l_2)}]_{q-l, q-l'} = [R^{(l_0 l_2)}]_{ll'}. \tag{7.4}$$

The matrices (7.2) have been constructed as solutions to the braid relations of proposition 3. They must also solve the phase relations of proposition 2. This yields the constraint

$$\eta^4 = (-\omega)^{-3}, \tag{7.5}$$

as well as the following equations for the dimensional trajectories $h_l = h([l])$

$$e(2h_{l+1} - 2h_l) = \eta^2(-\omega)^{l+2},$$
$$e(h_{l+1} - 2h_l + h_{l-1}) = \eta^2(-\omega)^2. \tag{7.6}$$

We may absorb the sign of the square root $\eta^2 = \pm(-\omega)^{-3/2}$ in the freedom $p \to p + q$ which leaves R unchanged, and assume $\eta^2 = (-\omega)^{-3/2}$. Still, we find two different trajectories according to $e(h_2 - h_1) = \pm(-\omega)^{3/4}$

$$e(h_l) = e\left(h_1 + \frac{(l^2-1)p - 4\varepsilon(l-1)q}{4q}\right), \tag{7.7}$$

where $\varepsilon = 0$, or $\varepsilon = \frac{1}{2}$.

They are easily identified with the Kac spectrum [9] $h_l = h_{1,l}(c(p,q))[h_1 = 0, \varepsilon = \frac{1}{2}; c(p,q) = 1 - 6(p-q)^2/pq]$

$$h_{1,l} = \frac{(l^2-1)p - 2(l-1)q}{4q}, \tag{7.8}$$

with the Kac spectrum $h_l = h_{k,l}(c(p,q))[h_1 \neq 0, \varepsilon = \frac{1}{2}k \bmod 1]$

$$h_{k,l} = \frac{(lp-kq)^2 - (p-q)^2}{4pq} \tag{7.9}$$

or with the SU(2) WZW spectrum $[h_1 = 0, \varepsilon = 0; l \equiv 2j+1, q \equiv k+2, p = 1]$

$$h_j = \frac{j(j+1)}{k+2}. \tag{7.10}$$

Observe that both the *unitary* minimal models [22] with $|p-q|=1$, and the WZW models with $p=1$ indeed have unitary exchange matrices.

For the minimal trajectories, the fusion rules (7.1) are those of the field $\phi_{(1,2)\cdot}$ along a horizontal row of the "BPZ rectangle" [9]. The usual identification of $V_{k,l}$ with $V_{p-k,q-l}$ is consistent with the symmetry (7.4). After a relabelling of l, the fusion rules (7.1) are also those of the field $\phi_{(1,q-2)\cdot}$ along a row of the BPZ rectangle (see subsect. 7.4). We shall discuss in subsect. 7.2 how, by strand formation (see above), the "horizontal" light-cone fields a_l associated with $\phi_{(1,l)\cdot}$ can be included in the exchange algebra, and in subsect. 7.3 how the minimal models can be completed by the introduction of a "vertical" elementary field b associated with $\phi_{(2,1)\cdot\cdot}$.

For the WZW trajectory, the fusion rules (7.1) are those of the doublet field $\phi_{2\cdot}$ with the isospin-($j=(l-1)/2$) multiplet fields $\phi_{l\cdot}$ which are primary with respect to the enlarged symmetry algebra. With respect to the Virasoro algebra, every isospin component $\phi_{ljm\cdot}$ ($l=1,2,\dots,k+1$; $j=(l-1)/2+0,1,\dots$; $m=-j,\dots,+j$) is primary with the fusion rules

$$[2,\tfrac{1}{2},\mu][l,j,m]=\bigoplus_{l'=l\pm1}\ \bigoplus_{j'=j\pm\frac{1}{2}}[l',j',m+\mu] \tag{7.11}$$

while their exchange algebra is given by R matrices (7.2) with the relevant l labels, but irrespective of j and m. The reader may convince himself that, e.g., the use of the *same* R matrix element in

$$P_{[1,j,0]}a_{(2,\frac{1}{2},+\frac{1}{2})}(x_1)\,a_{(2,\frac{1}{2},-\frac{1}{2})}(x_2)\,\Omega=[R^{(1,1)}]_{22}P_{[1,j,0]}a_{(2,\frac{1}{2},-\frac{1}{2})}(x_2)\,a_{(2,\frac{1}{2},+\frac{1}{2})}(x_1)\,\Omega$$

for both the singlet ($j=0$) and the triplet ($j=1$) Virasoro sectors within the vacuum ($l=1$) Kac–Moody sector is in perfect agreement with the relative sign expected from isospin (anti-)symmetrization as well as from the analytic behaviour of the 3-point functions with $h([1,0,0])=0$ and $h([1,1,0])=1$.

7.2. STRAND FORMATION

Let us concentrate on the minimal model interpretation of eq. (7.7). Assume $[1]=[\text{vacuum}]$, thus $[\alpha]=[2]$. We have given in eq. (7.2) only the exchange algebra of the elementary field $a=a_2$, associated with $\phi_{(1,2)\cdot\cdot}$. The operator product of this field with itself is known to produce fields $\phi_{(1,n+1)\cdot\cdot}$. Our idea is to construct the corresponding light-cone fields $(a_{n+1})_{ll'}$ as short-distance limits of n fields a_2

$$P_{l_0}a_{n+1}(x)P_{l_n}=\lim f(x,\partial_x)\sum_{l_1\dots l_{n-1}}c_{l_1\dots l_{n-1}}P_{l_0}a(x_1)P_{l_1}\dots P_{l_{n-1}}a(x_n)P_{l_n}. \tag{7.12}$$

Actually we do not need to calculate, but rather can take over Jimbo's et al. calculations [21]. The reason is that up to similarity transformations and some overall factor, our matrices $[R^{(l_0 l_2)}_{(22)}]_{ll'}$ coincide with the RSOS model Boltzmann weights [15]

$$W_{11}(l, l_0, l', l_2|u),$$

in the limit $u \to -i\infty$ at $K = \frac{1}{2}\pi$, $2\eta = \pi p/q$. The "fusion" of ref. [21] is equivalent to the projection onto the leading irreducible representation in the braid-group representation induced by strand formation à la sect. 5, or the construction of a_{n+1} as a linear combination of operator products with n fields a_2 à la sect. 6.

We quote the result

$$c_{l_1 \dots l_{n-1}} \sim \prod_{j=1}^{n-1} i^{l_j} s(l_j)^{-1/2}, \tag{7.13}$$

which can be made real by multiplication with some common power of i. The corresponding exchange matrices $R_{(n+1, n'+1)}$ of the composite fields a_{n+1} can be computed as the appropriate $u \to -i\infty$ limits of the (symmetrized) weights $W_{nn'}(l, l_0, l', l_2|u)$ of ref. [21].

The higher fusion rules [9]

$$[n_1][n_2] = \bigoplus_{n=|n_1-n_2|+1}^{\min(n_1+n_2-1, 2q-n_1-n_2-1)} [n], \tag{7.14}$$

where the sum leaves out every second term, can be recovered by combinatorial arguments.

7.3. THE EXCHANGE ALGEBRA OF THE COMPLETE MINIMAL MODELS

We have, so far, constructed the algebra of the field a associated with $\phi_{(1,2)}$. in the BPZ rectangle [9], and (in principle) that of the fields a_n associated with $\phi_{(1,n)}$. as well. The "structure constants" R of this algebra of fields interpolating among the sectors $V_{k,l}$ with k fixed (i.e. interpolating within the horizontal rows of the rectangle) are independent of k. Actually k came into eq. (7.9) by hand, choosing $h_1 = h_{k,1}$. In this sense we may speak of (a priori inequivalent) representations, labelled by k, of the same "horizontal" field algebra $\mathscr{A}$.

In the minimal models there is a "vertical" field b associated with $\phi_{(2,1)}$. outside the horizontal field algebra, interpolating among different k representations of $\mathscr{A}$. The field b in turn is the germ of a "vertical" field algebra $\mathscr{B}$ acting within the lth columns of the BPZ rectangle, while the horizontal field a interpolates among different l representations of $\mathscr{B}$. One may speak of a and b as "mutual soliton fields" intertwining between different "charged" representations of the field algebra of each other.

For the field algebra of the *complete* minimal models we only lack the exchange relations of a with b, which are clearly constrained by propositions 2 and 3.

Consider two fields a and b, both of the type described in subsect. 7.1, which we want to identify in the end with the light-cone fields associated with $\phi_{(1,2)}$. and $\phi_{(2,1)}$. of the BPZ minimal models. We thus assume the fusion rules of b not to "interfere" with those of a. In other words, there are sectors $V_{k,l}$, $1 \le k < q'$, $1 \le l < q$, and the fusion rules

$$[a][k,l] = [k,l+1] \oplus [k,l-1],$$
$$[b][k,l] = [k+1,l] \oplus [k-1,l]. \tag{7.15}$$

Introducing projectors $P_{k,l}$ onto $V_{k,l}$, and $P_{k.} = \Sigma_l P_{k,l}$, $P_{.l} = \Sigma_k P_{k,l}$, we have $aP_{k.} = P_{k.}a$ and $bP_{.l} = P_{.l}b$, while $P_{k,l\pm1}aP_{kl}$ satisfy an exchange algebra with structure constants (7.2) independent of k, and $P_{k\pm1,l}bP_{kl}$ satisfy an analogous exchange algebra independent of l. The former is parametrized by phases $-\omega = e(p/q), \eta$, the latter by phases $-\omega' = e(p'/q'), \eta'$.

Now the exchange algebra of a with b is of the form

$$P_{k\pm1l\pm l}a(x_1)P_{k\pm1l}b(x_2)P_{kl} = \chi_{kl}^{k\pm1l\pm1}P_{k\pm1l\pm1}b(x_2)P_{kl\pm1}a(x_1)P_{kl}. \tag{7.16}$$

One may absorb much ambiguity into rescalings $P_{k.}a \to \lambda(k)P_{k.}a$, $P_{.l}b \to \lambda(l)P_{.l}b$, which leave the algebras (7.2) unchanged, and solve the braid condition (proposition 3) by

$$\chi_{kl}^{k+1l+1} = \chi_{kl}^{k-1l-1} =: \chi,\ \chi_{kl}^{k+1l-1} = \chi_{kl}^{k-1l+1} =: \tilde{\chi}$$
$$\chi^2 = \tilde{\chi}^2, \tag{7.17}$$

and the phase condition (proposition 2) by

$$\tilde{\chi} = \chi^{-1}; \tag{7.18}$$

$\chi \to -\chi$ may again be absorbed in rescalings of $P_{k.}a$ and $P_{.l}b$. Thus, there are two essentially different solutions: $\chi = \tilde{\chi} = 1$ corresponding to the obvious tensor product $a \otimes b$ of two exchange field theories; and

$$\chi = -\tilde{\chi} = i. \tag{7.19}$$

The latter yields the complete dimensional trajectory

$$e(h_{kl}) = e\left(h'_k + h_l - \tfrac{1}{2}(k-1)(l-1)\right), \tag{7.20}$$

h'_k and h_l as in eq. (7.8), which gives the Kac spectrum (7.9) for $p' = q, q' = p$.

We lack an argument for $p' = q, q' = p$. Actually this cannot follow from the exchange algebra relations alone which do not exclude a decomposition of the Virasoro generators $L_n = \bigoplus_i L_n^{(i)}$. We must make an assumption concerning the uniqueness of the stress-energy tensor.

Denoting $\mathscr{T}_a$ (respectively $\mathscr{T}_b$) the $d=2$ quasiprimary fields in the vacuum sector occurring in the short-distance expansions of primary fields $a \cdot a$ (respectively $b \cdot b$), we require

$$\mathscr{T}_a = \mathscr{T}_b = \mathscr{T}. \tag{7.21}$$

In order to check this relation we compute the four-point functions $(\Omega, aaaa\Omega)$, $(\Omega, bbaa\Omega)$, $(\Omega, bbbb\Omega)$, and compare their short-distance limits $(\Omega, \mathscr{T}_a\mathscr{T}_a\Omega)$, $(\Omega, \mathscr{T}_b\mathscr{T}_a\Omega)$, $(\Omega, \mathscr{T}_b\mathscr{T}_b\Omega)$ which should have the common amplitude $\frac{1}{2}c$, where c is the "conformal anomaly" of the theory. The primary four-point functions are obtained from their monodromy properties which are in turn determined by the exchange algebras of the fields involved. We find

$$\begin{aligned}
&(\Omega, a(x_1)a(x_2)P_{1,1}a(x_3)a(x_4)\Omega) \\
&\quad = \left((x_1-x_2)(x_3-x_4)\right)^{-2h_2}(1-x)^{-2h_2}\,{}_2F_1\!\left(\frac{2-m}{m+1}, \frac{1}{m+1}; \frac{2}{m+1}; x\right), \\
&(\Omega, a(x_1)a(x_2)P_{1,3}a(x_3)a(x_4)\Omega) \\
&\quad = \alpha\left((x_1-x_2)(x_3-x_4)\right)^{-2h_2}(1-x)^{-2h_2}x^{h_3}\,{}_2F_1\!\left(\frac{1}{m+1}, \frac{m}{m+1}; \frac{2m}{m+1}; x\right), \\
&(\Omega, b(x_1)b(x_2)P_{1,1}a(x_3)a(x_4)\Omega) \\
&\quad = (x_1-x_2)^{-2h_2}(x_3-x_4)^{-2h_2'}(1-x)^{-1/2}\left(1-\tfrac{1}{2}x\right), \\
&(\Omega, b(x_1)b(x_2)P_{1,1}b(x_3)b(x_4)\Omega) \\
&\quad = \left((x_1-x_2)(x_3-x_4)\right)^{-2h_2'}(1-x)^{-2h_2'}\,{}_2F_1\!\left(\frac{2-m'}{m'+1}, \frac{1}{m'+1}; \frac{2}{m'+1}; x\right), \\
&(\Omega, b(x_1)b(x_2)P_{3,1}b(x_3)b(x_4)\Omega) \\
&\quad = \alpha'\left((x_1-x_2)(x_3-x_4)\right)^{-2h_2'}(1-x)^{-2h_2'}x^{h_3'}\,{}_2F_1\!\left(\frac{1}{m'+1}, \frac{m'}{m'+1}; \frac{2m'}{m'+1}; x\right),
\end{aligned} \tag{7.22}$$

where $m = p/(q-p)$,

$$\alpha = \sqrt{\Gamma\!\left(\frac{2m-1}{m+1}\right)\Gamma\!\left(\frac{m}{m+1}\right)\Gamma\!\left(\frac{2}{m+1}\right)\Gamma\!\left(\frac{1-m}{m+1}\right)\Big/\Gamma\!\left(\frac{2m}{m+1}\right)\Gamma\!\left(\frac{m-1}{m+1}\right)\Gamma\!\left(\frac{1}{m+1}\right)\Gamma\!\left(\frac{2-m}{m+1}\right)},$$

and similarly for $p/q \to p'/q'$; $x = (x_1-x_2)(x_3-x_4)/(x_1-x_3)(x_2-x_4)$.

The short-distance limits are computed according to sect. 6 with the normalizations of $\mathscr{T}_{a,b}$ fixed by the requirement that $-(1/2\pi)\int dx\, \mathscr{T}_{a,b}(x)$ generate translations on the fields a, b

$$P_{1,1}\mathscr{T}_a(x) = \lim_{x'\to x}\left(-\frac{m+3}{m}\partial_x\partial_{x'}(x-x')^{2h_2}P_{1,1}a(x)a(x')P_{1,1}\right), \quad (7.23)$$

and similarly for $\mathscr{T}_b(x)$. Then, we compute the amplitudes

$$c_{aa} = \frac{(m-2)(m+3)}{m(m+1)}, \qquad c_{bb} = \frac{(m'-2)(m'+3)}{m'(m'+1)},$$

$$c_{ba} = \frac{m+3}{m}\,\frac{m'+3}{m'},$$

which coincide iff $m + m' + 1 = 0$, i.e. $p'/q' = q/p$. Then

$$c = 1 - 6(p-q)^2/pq. \quad (7.24)$$

This completes the construction of (indecomposable) conformal exchange field theories with the full BPZ fusion rules. The theories we find are in fact the BPZ minimal models. If the horizontal and vertical exchange algebras are computed via strand formation as described in subsect. 7.2, with exchange matrices $[R^{(l_0l_2)}_{a(n_1n_2)}]_{ll'}$ (respectively $[R^{(k_0k_2)}_{b(m_1m_2)}]_{kk'}$) for light-cone fields a_n (respectively b_m) associated with $\phi_{(1,n)}$ (respectively $\phi_{(m,1)}$), then the exchange matrices of light-cone fields $c_{mn} =$ (short-distance limit of $b_m \cdot a_n$) associated with $\phi_{(m,n)}$ turn out to be

$$R^{(k_0l_0,\, k_2l_2)}_{(m_1n_1,\, m_2n_2)} = i^{(n_1-1)(m_2-1)+(n_2-1)(m_1-1)} I_{12} R^{(k_0k_2)}_{b(m_1m_2)} \otimes R^{(l_0l_2)}_{a(n_1n_2)} I_{21}^{-1} \quad (7.25)$$

where $I_{ij} = \mathrm{diag}((-1)^{(n_i-1)(k-k_2-m_j+1)/2}) \otimes \mathrm{diag}((-1)^{(m_j-1)(l-l_0+n_i-1)/2})$ are diagonal sign matrices which affect neither the braid nor the phase relations. It is the overall power of i in eq. (7.25) that causes the non-canonical dimension $h_{m,n} = h_{1,n} + h_{m,1} - \frac{1}{2}(m-1)(n-1)$; compare this with the remark in sect. 5.

Note that as a by-product we have computed the conformal anomaly from monodromy properties. This should be possible in general [6]. A less indirect path from the numerical structure constants R to the numerical value of c is, however, lacking.

7.4. LOCAL FIELDS IN MINIMAL MODELS

Let us now give some examples of local fields constructed with the light-cone fields of two, left and right, exchange algebras as building blocks. For the sake of transparency let us concentrate on the "horizontal" fields of subsect. 7.3.

We have started with symmetric exchange matrices $R^{(\cdot)} = R^{(\cdot)}_{(22)}$ of the "elementary" field $a = a_2$. Even without performing the explicit decomposition of the strand product this ensures that – with appropriate normalizations – all exchange matrices $R^{(\cdot)}_{(n_1 n_2)}$ of "composite" fields a_n are symmetric if $n_1 = n_2$, respectively

$$R^{(\cdot)}_{(n_1 n_2)} = \left[R^{(\cdot)}_{(n_2 n_1)} \right]^{\mathrm{T}}. \tag{7.26}$$

Hence, recalling the special case (1) of proposition 4, we conclude that

$$\phi_{(1,n)(1,n)}(x) = \sum_{ll'} (a_n^+)_{ll'}(x_+) \otimes (a_n^-)_{ll'}(x_-), \tag{7.27}$$

are parity invariant fields, local with respect to each other.

The symmetry (7.4) of the elementary R matrices entails the symmetry

$$\left[R^{(q-l_0, q-l_2)}_{(n_1 n_2)} \right]_{q-l, q-l'} = \left[R^{(l_0 l_2)}_{(n_1 n_2)} \right]_{ll'}, \tag{7.28}$$

of the composite R matrices. Hence, recalling the special case (2) of proposition 4 (with $i = id$), we conclude that

$$\tilde{\phi}_{(1,n)(1,n)}(x) = \sum_{ll'} (a_n^+)_{ll'}(x_+) \otimes (a_n^-)_{q-l, q-l'}(x_-), \tag{7.29}$$

are again parity invariant, mutually local fields. Actually $\tilde{\phi}$ interpolate only among sectors $V_{1,l} \otimes V_{1,q-l}$ and have no component acting on the vacuum $\Omega \in \mathrm{V}_{1,1} \otimes V_{1,1}$. Thus the correct local field in a model with asymmetric sectors should be $\phi_{(1,n)(1,n)} + \tilde{\phi}_{(1,n)(1,n)}$ rather than $\tilde{\phi}_{(1,n)(1,n)}$.

Finally, the fusion rules (7.14) of the composite fields tell that $[q-2][n] = [q-n+1] \oplus [q-n-1]$ which is equivalent to

$$[q-2][j(n)] = [j(n+1)] \oplus [j(n-1)], \tag{7.30}$$

where the 1 : 1 mapping j is defined by

$$\begin{aligned} j(n) &= q-n, \quad \text{if} \quad \min(n, q-n) = \text{even}, \\ j(n) &= n, \quad \text{if} \quad \min(n, q-n) = \text{odd}. \end{aligned} \tag{7.31}$$

Recall that eqs. (7.2) were *derived* from the fusion rules (7.1) [4]. Since a_{q-2} has isomorphic fusion rules (7.30), $[R^{(j(l_0), j(l_2))}_{(q-2, q-2)}]_{j(l)j(l')}$ must coincide with $[R^{(l_0 l_2)}_{(22)}]_{ll'}$ – possibly with different values $\tilde{p}$ and $\tilde{\eta}$. Inverting the argument by which we computed h_2 from η and p/q in subsect. 7.1, we now compute $\tilde{\eta}$ and $\tilde{p}/q$ from $h_{j(2)} = h_{q-2}$. We find $\tilde{p} = p$ and $\tilde{\eta} = \pm\eta$ (provided $h_2 = h_{q-2} \bmod \frac{1}{2}$, which is the case iff $p \cdot q =$ even, in particular for the unitary minimal models with $|p-q| = 1$).

Recalling the special case (2) of proposition 4 (with $i=j$) we conclude that

$$\phi_{(1,2)(1,q-2)}(x) = \sum_{ll'} (a_2^+)_{ll'}(x_+) \otimes (a_{q-2}^-)_{j(l),j(l')}(x_-), \tag{7.32}$$

is a commuting or anticommuting local field of spin $h_2 - h_{q-2} \in \frac{1}{2}\mathbb{Z}$, interpolating among symmetric and antisymmetric sectors $V_{1,l} \otimes V_{1,j(l)}$.

These constructions are strongly reminiscent of the field content of the diagonal and nondiagonal modular-invariant partition functions [23]. Some more care is needed for other non-parity-invariant fields $\phi_{(1,n)(1,q-n)}$, and for reasons why possibly the sums ((7.27), (7.29), (7.32)) should run over a subset of admissible values only.

Among all other minimal models the supersymmetric one [12] together with its superconformal generator of dimension $\frac{3}{2}$, $G(x) = a_{1,4}(x) \otimes \mathbf{1}$, as a "composite field" comes out as a special realization of our approach. It is remarkable that superfields can be constructed from "ordinary" exchange fields, and supersymmetry emerges without having been imposed.

7.5. ANALYTIC PROPERTIES OF n-POINT FUNCTIONS

The n-point functions $F_{\boldsymbol{\beta\alpha}} = (\Omega, a_{\alpha_1}(x_1)P_{\beta_1} \dots P_{\beta_{n-1}} a_{\alpha_n}(x_n)\Omega)$ may be considered [6] as local horizontal sections in complex vector bundles over $\mathring{\mathbb{C}}^n/\mathrm{S}_n$ with a flat connection, in which the representations of the braid group (proposition 5) describe the holonomy group of parallel transport. The matrices $\rho(b)$ act on the labels $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$. The action on $\boldsymbol{\alpha}$ is just the permutation $\pi(b) \in \mathrm{S}_n$.

The monodromy subgroup $\{\rho(b) | \pi(b) = e \in \mathrm{S}_n\}$ generated by $\rho(\sigma_j^2)$ describes the parallel transport in a flat connection of complex vector bundles over $\mathbb{C}^n$. The monodromy matrices act only on the labels $\boldsymbol{\beta}$ and describe the linear transformation of *all* conformal block functions, contributing to *one* local n-point function, into each other under analytic continuation of x_i around x_j.

The linear monodromy behaviour implies [24] that the conformal block functions are solutions to algebraic partial linear differential equations (PLDE's) of the Fuchs type. In general the latter are not uniquely specified by the monodromy, since quasiprimary fields share the monodromy properties of their primaries. Already Riemann [25] has discussed how the powers of $(x_i - x_j)$ near the singularities – i.e. the dimensions d_α and h_β involved – and the associated monodromy determine the PLDE's and their solutions.

In the minimal and WZW models, the PLDE's are obviously the equations inferred from the existence of degenerate states in the Verma modules [9, 11]. From the fusion rules (7.1) and the construction of proposition 5 one learns that every four-point function involving at least one "elementary" field $a = a_2$ has at most two "channels" P_{β_2}; i.e. the monodromy matrices are at most 2×2. Then the corresponding PLDE's – which can be converted into ordinary LDE's in the variable

$x=(x_1-x_2)(x_3-x_4)/(x_1-x_3)(x_2-x_4)$ – are of at most second order, and their solutions are hypergeometric functions, up to powers of x and $(1-x)$ and polynomial factors. By comparison with the known monodromy behaviour of hypergeometric functions [26], the conformal blocks can be read off their monodromy matrices, up to polynomial factors of degree increasing with the level of quasiprimary fields [19].

As an example we compute all primary four-point functions

$$(\Omega, a_{kl}(x_1)a_{12}(x_2)P_{k,l\pm1}a_{12}(x_3)a_{kl}(x_4)\Omega)$$
$$=: (x_1-x_4)^{-2h_{kl}}(x_2-x_3)^{-2h_{12}}x^{-h_{12}-h_{kl}}f_\pm(x). \qquad (7.33)$$

Eq. (3.10) and a similar equation for $R^{(0\beta)}$ provide the monodromy matrices $\rho_1^2=\rho_3^2$. ρ_2 is given by the exchange matrices (7.2). Thus the (positive-oriented) analytic continuation of x around 0 effects a factor of $e(h_{k,l\pm1})$ on $f_\pm(x)$, and the (positive-oriented) analytic continuation $x\to 1/x$ is described by

$$\begin{pmatrix} f_-\left(\frac{1}{x}\right) \\ f_+\left(\frac{1}{x}\right)\end{pmatrix} = e(h_{12})x^{-2h_{12}-2h_{kl}}R^{(ll)}\begin{pmatrix} f_-(x) \\ f_+(x)\end{pmatrix}. \qquad (7.34)$$

Taking into account the powers of the singularities at $x=0,1,\infty$ and comparing with the analytic behaviour $x\to 1/x$ of hypergeometric functions, we could derive

$$f_\pm(x)=c_\pm(1-x)^{h_{13}}x^{h_{k,l+1}}{}_2F_1\left(\frac{p}{q}, \pm k+(1\pm l)\frac{p}{q}; 1\pm k\mp l\frac{p}{q}; x\right), \qquad (7.35)$$

$$\frac{c_+}{c_-}=\sqrt{\gamma\left(k-l\frac{p}{q}\right)\gamma\left(1+k-l\frac{p}{q}\right)\gamma\left(-k+(l+1)\frac{p}{q}\right)\gamma\left(1-k+(l-1)\frac{p}{q}\right)}$$
$$\gamma(x):=\Gamma(x)/\Gamma(1-x). \qquad (7.36)$$

The same results have been obtained previously with different methods [27]. The above computation illustrates that our methods in principle are not weaker than others. There is the advantage – as compared with the solution of PLDE's by multiple contour integrals – that we work from the beginning with a natural basis $F_{\beta\alpha}$ of solutions, and have control over their analytic properties.

Let us now turn to an interesting side-remark [6] concerning the n-point functions of the "elementary" fields $a=a_2$ and the corresponding representations ρ of the braid groups B_n. One checks easily that all matrices (7.2) satisfy $R^2=\eta(1+\omega)R-$

$\eta^2\omega\mathbf{1}$. Introducing

$$g_i = (-\omega\eta)^{-1}\rho_i, \qquad t = (-\omega)^{-1}, \qquad e_i = (1+g_i)/(1+t), \tag{7.37}$$

this entails

$$g_i^2 + (1-t)g_i - t = 0, \tag{7.38}$$

or, equivalently

$$e_i^2 = e_i. \tag{7.39}$$

These are – besides the representation conditions of the braid group – the defining relations of the Hecke algebra $\mathrm{H}_n(t)$ [28] generated by g_i. The projectors e_i are *real* in the case of *unitary R* matrices.

Actually the Hecke algebra relations just tell that the generators g_i, and thus ρ_i, have only two different eigenvalues. This property, in turn, just reflects the two-fold branching of the fusion rules (7.1). Consequently, the exchange algebra of composite fields will generally not give rise to "Hecke-type" representations of B_n.

Next, it is easy to verify that the projectors e_i satisfy

$$e_i e_{i\pm1} e_i = \tau e_i,$$

$$\beta \equiv \tau^{-1} = 4\cos^2(\pi p/q). \tag{7.40}$$

These are – besides the Hecke algebra relations – the defining relations of the Jones algebras $\mathrm{A}_{\beta,n}$ [29]. Our unitary cases $\beta = 4\cos^2(\pi/q)$ have been identified previously [29] as the only β values (except $\beta = 4$) for which unitary "Jones-type" representations of the braid groups exist.

Moreover, Jones has classified [29] the cases in which these representations define *finite* matrix groups. In our context this is equivalent to the property of the n-point functions to be *algebraic* functions. The result is that for $q = 3, 4, 6, 10$ the n-point functions (for $q = 10$ only $n \leq 4$) of the elementary fields are algebraic functions.

$q = 3$ corresponds to the vertical field $\phi_{(21)\cdot}$ $(h = \frac{1}{2})$ of the Ising model $(c = \frac{1}{2})$. $q = 4$ corresponds to the horizontal field $\phi_{(12)\cdot}$ $(h = \frac{1}{16})$ of the Ising model, and to the vertical field $\phi_{(21)\cdot}$ of (the universality class of) the tricritical Ising model $(c = \frac{7}{10})$. $q = 6$ corresponds to $\phi_{(12)\cdot}$ in (the universality class of) the three-state Potts model $(c = \frac{4}{5})$, and to $\phi_{(21)\cdot}$ in (the universality class of) the tricritical three-state Potts model $(c = \frac{6}{7})$. $q = 10$ is realized in the models with $c = \frac{14}{15}$ and $c = \frac{52}{55}$.

The Ising model functions are well known. For $c = \frac{4}{5}$, $h_{12} = \frac{1}{8}$, we have computed from the analytic exchange behaviour of

$$\left(\Omega, a_{12}(x_1)a_{12}(x_2)\underset{11}{P_{13}}a_{12}(x_3)a_{12}(x_4)\Omega\right) = \prod_{i<j}(x_i - x_j)^{-1/12} f_{\pm}(x), \tag{7.41}$$

the algebraic functions

$$f_+(x) = c' j(x)^{1/12} \sqrt{\sum_{e^3=1} (-e)\sqrt{1 - ej(x)^{-1/3}}} = c_+ x^{1/2}(1 + \mathrm{O}(x))$$

$$f_-(x) = c' j(x)^{1/12} \sqrt{\sum_{e^3=1} \sqrt{1 - ej(x)^{-1/3}}} = c_- x^{1/6}(1 + \mathrm{O}(x))$$

$$\frac{c_+}{c_-} = \sqrt{\frac{\Gamma(-\frac{2}{3})\Gamma(\frac{1}{3})\Gamma(\frac{3}{2})\Gamma(\frac{5}{6})}{\Gamma(\frac{5}{3})\Gamma(\frac{2}{3})\Gamma(-\frac{1}{2})\Gamma(\frac{1}{6})}} = 3 \times 2^{-13/6}, \tag{7.42}$$

where the sums extend over the three cubic roots of unity, and

$$j(x) = \tfrac{4}{27}(1 - x + x^2)^3 / x^2(1 - x)^2,$$

is a familiar invariant of the theory of elliptic ϑ-functions [30].

By eq. (7.35), the four-point functions of a_{12} for all minimal models are expressed as hypergeometric functions. The cases $q = 3, 4, 6, 10$, $p = q \pm 1$, precisely fit into the famous Schwarz list [31] of *algebraic* hypergeometric functions. To our knowledge, the explicit formulae

$$_2F_1(-\tfrac{1}{2}, \tfrac{1}{6}; \tfrac{1}{3}; x) = (1 - x + x^2)^{1/4} \sqrt{\tfrac{1}{3} \sum_{e^3=1} \sqrt{1 - ej(x)^{-1/3}}},$$

$$3 \times 2^{-13/6} x^{2/3} {}_2F_1(\tfrac{5}{6}, \tfrac{1}{6}; \tfrac{5}{3}; x) = (1 - x + x^2)^{1/4} \sqrt{\tfrac{1}{3} \sum_{e^3=1} (-e)\sqrt{1 - ej(x)^{-1/3}}}, \tag{7.43}$$

are not displayed in the handbooks on hypergeometric functions.

7.6. COUNTEREXAMPLE

We have solved the braid relations (3.11) for a field $a = a_4$ with the following assumed fusion rules ([1] = [vacuum])

$$[4][i] = [4], \qquad i = 1, 2, 3,$$

$$[4][4] = [1] \oplus [2] \oplus [3], \tag{7.44}$$

finding

$$R^{(ii)} =: \eta\,, \qquad i = 1,2,3\,,$$

$$R^{(ij)} =: \eta\omega\,, \qquad i \neq j,\ = 1,2,3\,,$$

$$[R^{(44)}]_{ij} = \tfrac{1}{3}\eta \begin{pmatrix} 2\omega+1 & \omega-1 & \omega-1 \\ \omega-1 & 2\omega+1 & \omega-1 \\ \omega-1 & \omega-1 & 2\omega+1 \end{pmatrix},$$

$$\omega = e\left(\pm\tfrac{1}{3}\right). \tag{7.45}$$

The phase relation (3.9) can *not* be satisfied with $R^{(44)}$, indicating that eq. (7.45) cannot be associated with a *conformal* exchange algebra in the sense of this paper. Still this solution may have a relevance for the more abstract approach to local QFTh_2 beyond CQFTh_2 suggested in sect. 5.

The counterexample shows that proposition 2 not only constrains the dimensional spectrum in terms of the R matrices, but in fact constrains the exchange matrices themselves, and thus the fusion rules compatible with conformal invariance.

8. Conclusion and outlook

We presented the concept of a light-cone field theory with a non-commutative field algebra: the exchange algebra. Exchange field theory is regarded as a building block of non-perturbative interacting local QFTh_2. We discussed exchange field theory in much detail in the specific context of conformal QFTh_2 as a most natural – and for the moment the only available – realization. We outlined the strategies for a conclusive analysis of CQFTh_2 with finite fusion rules.

Although we expect that many of the ideas, if properly generalized, may transcend conformal and even two-dimensional QFTh [32], we made no real attempt at generalization in this paper. Furthermore, we leave to a future publication all aspects of representation theory of conformal exchange algebras related with KMS [3] temperature states. Such aspects will be useful [32] for the concept of "soliton completeness" and the reduction of so-called modularity properties of partition functions [23] to causality and completeness principles.

We would like to view exchange algebras as "the liberation of the very subtle ideas of Yang, Baxter [14], and Faddeev [33] from their narrow confinement to special lattice models". We have given arguments that (suitably generalized to field algebras with infinite families of quantum fields) the braid identities are properties of Einstein-causal CQFTh_2 "par excellence". As an extension of the recent work of Karowski [34] on the use of the algebraic Bethe-ansatz technique to relate the RSOS models via a θ-angle with the six-vertex models (or to relate minimal models with the Coulomb gas (respectively Sine–Gordon) field theory), one should study in our algebraic language the field-theoretical transmutation of exchange algebras with Burau braid representations into those of the minimal models. This may well also lead to a "liberation" of the Bethe-ansatz technique from special statistical mechanics. We have in mind a yet unknown use of the Bethe ideas (with completely different physical content) for the systematic construction of (vacuum) representations of exchange field algebras.

The immediate area of application of $d = 2$ CQFTh's is of course the understanding of the universality classes of surfaces (thin films, layers) of critical systems of condensed matter. It is remarkable (perplexing) that nature reveals its deepest causality secrets in an area whose physical principles are not directly related to Einstein's causality.

Note added

After completion of this work we became aware of ref. [35]. In their search for invariant link polynomials these authors discuss very systematically the limiting process leading from Yang–Baxter to braid matrices, as well as (in our terminology:) the projection onto braid-group representations of composite fields out of strand products of elementary representations.

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