Nuclear Physics B295 [FS21] (1988) 229-242
North-Holland, Amsterdam

QUASIPRIMARY FIELDS:
An approach to positivity of 2D conformal quantum field theory

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Received 11 August 1987

Among the "secondary fields" of Belavin, Polyakov, and Zamolodchikov, the quasiprimary fields are distinguished by their covariant behaviour under infinitesimal Möbius transformations. Local n-point functions can be described in terms of the numerical amplitudes of 3-point functions of quasiprimary fields. The positivity of the full field theory is guaranteed if certain nonlinear consistency conditions among these constants can be satisfied by real numbers. Constraints arising from these conditions are discussed.

1. Introduction

The understanding of two-dimensional conformally invariant quantum field theory has considerably advanced since the work of Belavin, Polyakov and Zamolodchikov [1]. The new, and by now very familiar techniques are most powerful for the construction of euclidean correlation functions. However, the "zeroth Wightman axiom" fell somewhat out of reach: the requirement that field operators are represented on a Hilbert space with positive definite norm. Friedan, Qiu and Shenker [2] have found necessary, and very restrictive conditions for the admissible representations of the energy-momentum tensor field; but the situation remains unclear for other fields occurring in the theory.

In this paper we shall introduce an algorithm for the study of positivity in conformally invariant theories. The algorithm applies directly to Wightman distributions in the Minkowski region. For a good understanding it is necessary to have a clear view of the global aspects of Minkowski quantum field theory in contradistinction to the local, euclidean point of view [3, 1].

The most important issue in our "global" approach is the transformation law of fields under global conformal (Möbius) transformations $\mathrm{SL}(2,\mathbb{R})$, and more generally under $\widetilde{Diff}_{\mathrm{cent}}(\mathbb{R})$, the central extension of the covering group of circular diffeomorphisms, and its consistency with Einstein causality. This transformation law involves a decomposition of local Minkowski fields into nonlocal parts which pick up different complex phases under the action of a central element of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ (sect. 2). For these irreducible parts there exist global operator product expansions,

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which in general involve (with the exception of canonical operators such as the energy-momentum tensor) integration over the whole light-cone (sect. 3). Unlike Wilson expansions, global expansions take a simple form only if expressed in terms of the nonlocal fields. They pose no convergence problems.

In contrast, the Kac-Feigin-Fuchs-BPZ approach [3, 1] is based on the study of euclidean fields and their derivatives at a special analytic point, which may be either the point $u = i$ in the upper halfplane (light-cone formulation) or the point $z = 0$ of the compact (circular) quantization. Global expansion techniques which involve integration over the entire light-cone are not used in this "local" approach. The existence of nonlocal fields with two quantum numbers (the scale dimension and the phase) would be difficult to see in this framework.

Let us, before continuing the schedule of this paper, clarify the distinction between operator aspects and the so-called euclidean field theory (the analytically continued correlation functions of statistical mechanics). For example the formal symmetry group $\mathrm{SL}(2,\mathbb{C})$ of the euclidean theory is not the quantum mechanical symmetry group of local conformal field theory. The latter is $\widetilde{\mathrm{SL}(2,\mathbb{R})}_{\mathrm{C}}$, the complexification of the universal covering of the Möbius group [4, 5]. This is only a semi-group (represented by non-unitary operators), acting on the enlarged domain of analyticity of local Wightman functions which has a complicated structure interweaving both light-cones.

Most of the contour manipulations done on analytically continued correlation functions can be derived from operator relations, but there is no general equivalence. Certain euclidean points as $z = 0$ in Kac-Feigin-Fuchs representation theory have an operator formalism attached to them: due to the spectrum condition local states at $u = i$ (light-cone picture) or $z = 0$ (compact picture) are exponentially damped and highly normalizable. Since arbitrary contour integrations cannot be done on exponentially damped states only, such manipulations are meaningless on operators and states. For a recent more detailed presentation of the operator approach we refer to ref. [5].

In sect. 4 we show, how the global operator product expansions can be used in order to derive transformation laws of quasiprimary fields under general diffeomorphisms of the light-cone.

In sect. 5 we exploit the general structure of 4-point functions implied by the global operator expansions valid on the vacuum. We establish a necessary and sufficient criterion for the positivity of the Hilbert space representation of local fields. This is evaluated recursively for the energy momentum tensor field in sect. 6, reproducing the results of ref. [2] in a different, field theoretical language.

For the study of representations of general fields one needs to understand the correct phase prescriptions of global operator product expansions valid between arbitrary states. The relation of this question with a recent insight into algebraic properties (which we called "exchange algebra") of light-cone fields [6] is sketched in sect. 7.

2. Transformation laws under Möbius transformations

The group of conformal transformations of two-dimensional space-time is the direct product of two Möbius groups $SL(2,\mathbb{R})_{L,R}$ acting on either light-cone. We denote by $u := x + t$, $v := x - t$ the light-cone variables. The element $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R})_L$ acts on u as

$$u \to f_A(u) = \frac{\alpha u + \beta}{\gamma u + \delta}.$$

Let

$$A_{\text{scale}} = \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}, \qquad A_{\text{trans}} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \qquad A_{\text{spec}} = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

denote scale transformations, translations, and special conformal transformations respectively. There is a compact subgroup $U(1) = \left\{ \begin{pmatrix} \cos\frac{1}{2}\phi & \sin\frac{1}{2}\phi \\ -\sin\frac{1}{2}\phi & \cos\frac{1}{2}\phi \end{pmatrix} \right\}$ which is generated by the "conformal hamiltonian" $H = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ such that $e^{-2\pi i H} = -\mathbb{1}$ takes u into itself.

A *conformal* field $\phi(u, v)$ transforms under an element of $SL(2,\mathbb{R})_L$ like a covariant tensor of anomalous dimension:

$$U(A)\phi(u,v)U^{-1}(A) = \left(\frac{\mathrm{d}f_A}{\mathrm{d}u}\right)^d \phi(f_A(u), v) \tag{1}$$

and similarly for the other light cone. The respective exponents $d, \bar{d}$ are the light-cone scale dimensions of the field $\phi = \phi_{d\bar{d}}$.

The transformation law (1) is meaningful only if $\mathrm{d}f_A/\mathrm{d}u = (\gamma u + \delta)^2 > 0$. It has to be completed by a specification of the complex phase beyond the singular point. It has been observed [7] that an irreducible transformation law like (1) for *local* fields is in conflict with the Einstein causality principle.

The solution to this apparent problem [8] comes from the fact that the local field $\phi_{d\bar{d}}$ does not transform homogeneously under finite Möbius transformations. There is a spectral decomposition

$$\phi_{d\bar{d}}(x) = \sum_{\xi\bar{\xi}} \phi_{d\bar{d}}^{\xi\bar{\xi}}(x) \qquad (\xi, \bar{\xi} \in \mathbb{R} \bmod \mathbb{Z}),$$

with respect to the nonlocal Hilbert space operators $Z = e^{-2\pi i L_0} \in \widetilde{SL(2,\mathbb{R})}_L$, $\bar{Z} = e^{2\pi i \bar{L}_0} \in \widetilde{SL(2,\mathbb{R})}_R$, which are minimal central elements of the covering groups representing the trivial light-cone transformations $e^{-2\pi i H} \in SL(2,\mathbb{R})_L$, $e^{2\pi i \bar{H}} \in SL(2,\mathbb{R})_R$. By definition, the *nonlocal parts* $\phi_{d\bar{d}}^{\xi\bar{\xi}}$ transform homogeneously under Z, $\bar{Z}$:

$$Z\phi_{d\bar{d}}^{\xi\bar{\xi}}(u,v)Z^{-1} = e^{-2\pi i(d-\xi)}\phi_{d\bar{d}}^{\xi\bar{\xi}}(u,v). \tag{2}$$

(Here and in the following, the respective formulae for the v light-cone are obtained by replacing $u \to v$; $i \to -i$; $d, \xi, Z \ldots \to \bar{d}, \bar{\xi}, \bar{Z} \ldots$). For a free field, the above spectral decomposition yields just two parts: the creation part ($\xi = 0$) and the annihilation part ($\xi = 2d$). The nonlocal parts $\phi_{d\bar{d}}^{\xi\bar{\xi}}$ of interacting fields are thus subtle generalizations of the creations and annihilation parts of free fields. Like the latter they have no euclidean analogue. In the case $c < 1$ which we are interested in (see below) the spectra of L_0, $\bar{L}_0$ are discrete, and the number of nonlocal parts of a local field is finite.

Now, the complex phases in the Möbius transformation law (1) are specified for the nonlocal parts by

$$U(A_{\text{scale}})\phi_{d\bar{d}}^{\xi\bar{\xi}}(u,v)U^{-1}(A_{\text{scale}}) = \lambda^d \phi_{d\bar{d}}^{\xi\bar{\xi}}(\lambda u, v),$$

$$U(A_{\text{trans}})\phi_{d\bar{d}}^{\xi\bar{\xi}}(u,v)U^{-1}(A_{\text{trans}}) = \phi_{d\bar{d}}^{\xi\bar{\xi}}(u+\beta, v),$$

$$U(A_{\text{spec}})\phi_{d\bar{d}}^{\xi\bar{\xi}}(u,v)U^{-1}(A_{\text{spec}}) = (1+\gamma u)^{-(2d,\xi)} \phi_{d\bar{d}}^{\xi\bar{\xi}}\left(\frac{u}{1+\gamma u}, v\right),$$

where

$$(1+\gamma u)^{-(2d,\xi)} := (1+\gamma(u+i\varepsilon))^{-2d+\xi}(1+\gamma(u-i\varepsilon))^{-\xi}. \tag{3}$$

There is no complex phase for the conformal transformations described by A_{scale} and A_{trans} which correspond to Minkowski scale and Poincaré transformations.

Following ref. [8], the spectrum condition implies, that on the ket vacuum $|0\rangle$ only $\phi_{d\bar{d}}^{00}$ can be non-zero, and on the bra vacuum $\langle 0|$ only $\phi_{d,\bar{d}}^{2d,2\bar{d}}$. From the invariance of the vacuum under Z we see, that $\phi_{d\bar{d}}|0\rangle = \phi_{d\bar{d}}^{00}|0\rangle$ are eigenstates of Z with eigenvalue $e^{-2\pi i d}$. The Hilbert space may be decomposed into sectors $\mathscr{H}_{h\bar{h}}$ on which $Z, \bar{Z}$ take values $e^{-2\pi i h}, e^{2\pi i \bar{h}}$ where $h, \bar{h}$ exhaust modulo $\mathbb{Z}$ the spectrum of scale dimensions of conformal fields.

In a local 3-point function $\langle \phi_{d_1\bar{d}_1}\phi_{d_2\bar{d}_2}\phi_{d_3\bar{d}_3}\rangle$ only the part $\phi_{d_2\bar{d}_2}^{\xi\bar{\xi}}$ with $\xi = d_2 + d_3 - d_1 \bmod \mathbb{Z}$ can contribute.

The most important progress [1] in two-dimensional conformal field theory is due to the realization that *primary* fields $\phi_{h\bar{h}}$ must exist, which transform like covariant tensors even under general (infinitesimal) diffeomorphisms of the light-cone. The states created by such fields from the vacuum are highest-weight states for the irreducible representation vectors $\mathscr{H}_{h\bar{h}}$ in Hilbert space of the energy-momentum tensor fields $T(u), \bar{T}(v)$.

However, primary fields cannot exhaust the field content of a theory. Instead, short-distance operator product expansions (Wilson expansions) of a primary field $\phi_{h\bar{h}}$ with the energy-momentum tensor will generate an infinite number of secondary fields ("conformal family"). A basis of these is an infinite set of *quasiprimary* fields

$\phi_{d\bar{d}}$ of dimensions $d \in h + \mathbb{N}_0$, $\bar{d} \in \bar{h} + \mathbb{N}_0$, and their derivatives. By definition a field is quasiprimary if it transforms like a covariant tensor under Möbius transformations. Under general diffeomorphisms the quasiprimary fields of one conformal family are transformed among each other; the "mixing coefficients" of such transformations laws must contain the Schwarz derivative $Df = f'''/f' - \frac{3}{2}(f''/f')^2$ of the diffeomorphism, which vanishes if and only if f is a Möbius transformation (see sect. 4).

The most familiar quasiprimary field is the energy-momentum tensor component $T(u)$ with the transformation law

$$T(u) \to f'(u)^2 \cdot T(f(u)) + \tfrac{1}{12} c (Df)(u) \cdot 1,$$

where the identity operator 1 is the primary field ϕ_{00} of the conformal family of the quasiprimary field $T \sim \phi_{20}$.

The parameter c is the conformal anomaly parameter. It plays a crucial role in the classification theory of conformally invariant quantum field theories. In particular, if the number of primary fields is finite, then c must be less than one [10], and if $c < 1$, then c can take only values in the discrete series $c(m) = 1 - (6/m(m+1))$, $m = 2, 3, 4, \dots$ [2]. In this paper we shall concentrate only on this case.

3. Global operator product expansions

For operator products of conformal fields, global expansions exist and may be discussed as group-theoretical expansions in euclidean Green functions [11] or of bilocal states [12]. In Minkowski space it is most convenient to work with the global expansions in terms of nonlocal parts of local fields [8]

$$\phi^{\xi\bar{\xi}}_{d_1\bar{d}_1}(x_1)\phi_{d_2\bar{d}_2}(x_2)|0\rangle = \sum_{d_3\bar{d}_3} 2\,e^{i\pi(d_3-\bar{d}_3)} c_{312} \int d^2x_3\, K(d_i, \bar{d}_i; x_i)\phi_{d_3\bar{d}_3}(x_3)|0\rangle \quad (4)$$

where the sum extends over fields with dimensions $d_3 = d_1 + d_2 - \xi \bmod \mathbb{Z}$, $\bar{d}_3 = \bar{d}_1 + \bar{d}_2 - \bar{\xi} \bmod \mathbb{Z}$.

For the expansion of local fields, the sum over all values of ξ, $\bar{\xi}$ must be taken.

Such expansions are valid on the vacuum only, since the spectrum condition (analyticity in the upper complex halfplane of the time coordinate of x_2) is essential for the existence of a well-defined integral kernel. The kernel K is uniquely determined by the transformation laws (3):

$$K(d_i, \bar{d}_i; x_i) = K(d_i, u_i) \cdot \bar{K}(\bar{d}_i, v_i),$$

$$K(d_i, u_i) = \frac{(2\pi)^{-1}\Gamma(2d_3)\Gamma(\lambda_3)\Gamma(1-\lambda_2)^{-1}}{(u_1 - u_2 - i\varepsilon)^{\lambda_1}(u_1 - u_3 + i\varepsilon)^{\lambda_2 - 1 + 2d_3}(u_1 - u_3 - i\varepsilon)^{1-2d_3}(u_2 - u_3 + i\varepsilon)^{\lambda_3}}, \quad (5)$$

with $\lambda_1 = d_1 + d_2 + d_3 - 1$, $\lambda_2 = d_1 - d_2 - d_3 + 1$, $\lambda_3 = -d_1 + d_2 - d_3 + 1$, and a similar formula ($u \to v$, $d \to \bar{d}$, $i\varepsilon \to -i\varepsilon$) for $\bar{K}(\bar{d}_i, v_i)$.

We orthonormalize the hermitian fields such that their 2-point functions are the following distributions giving rise to localized states of positive norm square:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \delta_{(\phi_1=\phi_2=\phi_{d\bar{d}})} \frac{e^{-i\pi(d-\bar{d})}}{(u_1-u_2-i\varepsilon)^{2d}(v_1-v_2+i\varepsilon)^{2\bar{d}}}. \tag{6}$$

Then the coefficients c_{312} in eq. (4) turn out to be the amplitudes of 3-point functions: if eq. (4) is multiplied by a bra state $\langle 0|\phi_{d_3\bar{d}_3}(x_3)$, the u_3 integration path can be closed in the upper halfplane, exploiting the spectrum condition. For $u_1 > u_2$ it passes along the real interval (u_2, u_1) below, and back above the real axis. The two pieces differ only by their complex phases, the difference of which contributes a factor of $2i \sin \pi\lambda_3 = 2\pi i/\Gamma(\lambda_3)\Gamma(1-\lambda_3)$ to an integral extending from u_2 to u_1. This integral yields a particular hypergeometric function [13] that reduces after cancellation of all, possibly singular Γ factors to the usual power-law form of the 3-point function:

$$\langle \phi_{d_3\bar{d}_3}(x_3)\phi_{d_1\bar{d}_1}(x_1)\phi_{d_2\bar{d}_2}(x_2)\rangle$$

$$= \frac{c_{312}}{(u_3-u_1-i\varepsilon)^{d_3+d_1-d_2}(u_3-u_2-i\varepsilon)^{d_3+d_2-d_1}(u_1-u_2-i\varepsilon)^{d_1+d_2-d_3}(v)}, \tag{7}$$

where (v) indicates similar factors referring to the v light-cone.

Now suppose that one of the fields contributing to the expansion (4) creates "ghost states" of negative norm square from the vacuum, thus violating the zeroth Wightman axiom. Its 2-point function then differs from the normalization (6) by a $(-)$ sign, which changes the 3-point function (7) by a $(-)$ sign. We prefer however to have eq. (7) valid for all fields, i.e. c_{312} should always be the amplitude of a 3-point function. Thus we must introduce a factor of (-1) into the expansion (4) for every field violating positivity. Of course, in an admissible quantum field theory such fields do not occur. We return to this point in sect. 6.

Since $T(u)$ transforms the fields $\{\phi_{h+n,\bar{d}} | \bar{d} \text{ fixed}, n \in \mathbb{N}_0\}$ irreducibly among each other and independently of $\bar{T}(v)$, the coefficients in (4) and thus the amplitudes c_{312} factorize as follows

$$c_{312} = c_{312}^{\text{prim}} \cdot N_{312}^{n_3 n_1 n_2} \cdot \bar{N}_{312}^{\bar{n}_3 \bar{n}_1 \bar{n}_2},$$

where the 3-point amplitude c_{312}^{prim} of the corresponding primary fields depends only on the respective conformal families, while $N_{312}^{n_3 n_1 n_2}$ and $\bar{N}_{312}^{\bar{n}_3 \bar{n}_1 \bar{n}_2}$ are relative amplitudes within the families for fields of dimensions $d_i = h_i + n_i$, $\bar{d}_i = \bar{h}_i + \bar{n}_i$. The notational distinction between h, the dimension of a primary field, and $d = h + n$, the dimension of a quasiprimary field in the same family, will be made throughout this paper.

From locality of the fields (6) with respect to each other, one concludes by complex conjugation and analytic continuation of 3-point functions that $\eta c_{ijk}^{\text{prim}}$ (where $\eta = 1$ resp. $\eta = i$ if among ϕ_i, ϕ_j, ϕ_k there are no resp. two Fermi fields) and $N_{ijk}^{n_i n_j n_k}$, $\bar{N}_{ijk}^{\bar{n}_i \bar{n}_j \bar{n}_k}$ are all *real* numbers. Moreover the symmetry

$$N_{kji}^{n_k n_j n_i} = (-1)^{n_i + n_j + n_k} N_{ijk}^{n_i n_j n_k}$$

is valid. These properties are crucial for the discussion of the positivity requirement for Wightman distributions, sect. 6.

4. Quasiprimary transformation laws under diffeomorphisms

This section is not necessary for the rest of the paper. We want to include it here as an illustration, how the transformation laws of quasiprimary fields under diffeomorphisms other than Möbius transformations can be derived from the operator product expansions (4).

Infintesimal conformal transformations of conformal fields are generated by the energy-momentum tensor, the light-cone components of which are themselves conformal fields. We shall thus consider the expansion (4) for $\phi_1 = T$, in which case only fields ϕ_3 of the conformal family of ϕ_2 contribute. We have $h_2 = h_3 =: h$, and $\bar{d}_2 = \bar{d}_3 =: \bar{d}$. We shall use the abbreviation $(-1)^m \sqrt{\frac{1}{2}c}\, N_{h0h}^{m2n} =: N_{mn} = N_{nm}$ for the coefficients occurring. Performing the same deformations of the paths of integration as in the preceding section, we obtain formally

$$\begin{aligned} &T(u_1)\phi_{h+n,\bar{d}}(u_2, v_2)|0\rangle \\ &\quad = \sum_{m=0}^{\infty} N_{mn} \frac{\Gamma(2h+2n)}{\Gamma(2h+m+n-2)\Gamma(2-n+m)} (u_1 - u_2)^{m-n-2} \\ &\qquad \times \int_0^1 \mathrm{d}t\, t^{1-n+m}(1-t)^{2h+n+m-3} \phi_{h+m,\bar{d}}(u_2 + t(u_1 - u_2), v_2)|0\rangle . \end{aligned}$$

Expanding $\phi_{h+m,\bar{d}}$ into a Taylor series around u_2, and cancelling the divergences of the t-integrals with the divergent Γ-factors, we obtain the Wilson expansion (suppressing the trivial v dependence from now on):

$$\begin{aligned} T(u_1)\phi_{h+n}(u_2)|0\rangle &= \left((h+n)(u_1-u_2)^{-2}\phi_{h+n}(u_2) + (u_1-u_2)^{-1}\partial_u \phi_{h+n}(u_2)\right)|0\rangle \\ &\quad + \sum_{m=0}^{n-2} N_{mn} \sum_{k=0}^{n-m-2} \frac{\Gamma(2h+2m)}{\Gamma(2h+2m+k)} \\ &\quad \times \binom{n-m-2}{k} (u_1-u_2)^{m-n-2+k} (-\partial_u)^k \phi_{h+m}(u_2)|0\rangle \\ &\quad + (\text{regular terms at } u_1 \approx u_2) . \end{aligned}$$

The commutator with $(1/2\pi i)\int_{-\infty}^{\infty} T(u_1)\varepsilon(u_1)\,du_1$ is evaluated by means of different $i\varepsilon$ boundary values: $(u-i\varepsilon)^{-l-1}-(u+i\varepsilon)^{-l-1}=(2\pi i/l!)(-\partial_u)^l\delta(u)$. The infinitesimal transformation law of the local field is the same as that of its vacuum nonlocal part. Hence under an infinitesimal diffeomorphism $u \to u+\varepsilon(u)$

$$\delta_\varepsilon \phi_{h+n}(u) = (h+n)\,\partial_u\varepsilon(u)\cdot\phi_{h+n}(u) + \varepsilon(u)\cdot\partial_u\phi_{h+n}(u)$$
$$+\sum_{m=0}^{n-2} N_{mn} \sum_{l=0}^{n-m-2} \frac{\Gamma(2h+2m)}{\Gamma(2h+m+n-l-2)} \binom{n-m-2}{l}$$
$$\times \frac{\partial_u^{l+3}\varepsilon(u)}{(l+3)!}(-\partial_u)^{n-m-2-l}\phi_{h+m}(u).$$

With the help of the composition law $g\circ f$ of diffeomorphisms we have integrated these variations to finite transformation laws under $u\to f(u)$ for $\phi_{h+n}(u)=T(u)$, and for normalized (eq. (6)) quasiprimary fields t_4 and t_6 of dimension 4 and 6 in the 1-family, and for a generic field $\phi_{h+2}(u)$:

$$T(u)\to f'^2\cdot T(f(u)) + \tfrac{1}{12}cDf\cdot 1,$$

$$t_4(u)\to f'^4\cdot t_4(f(u)) + \alpha f'^2\cdot Df\cdot T(f(u)) + \tfrac{1}{12}c\cdot\tfrac{1}{2}\alpha(Df)^2\cdot 1,$$

$$t_6(u)\to f'^6\cdot t_6(f(u)) + \beta f'^4\cdot Df\cdot t_4(f(u)) + \tfrac{1}{2}\alpha\beta f'^2\cdot(Df)^2\cdot T(f(u))$$
$$+\gamma\Big[\big(f'^2\cdot(Df)''-5f''\cdot f'\cdot(Df)'+5f''^2\cdot Df\big)\cdot T(f(u))$$
$$+\big(-\tfrac{5}{2}f'^3\cdot(Df)'+5f''\cdot f'^2\cdot Df\big)\cdot T'(f(u)) + f'^4\cdot Df\cdot T''(f(u))\Big]$$
$$+\tfrac{1}{12}c\Big[\big(\tfrac{1}{6}\alpha\beta-\tfrac{2}{3}\gamma\big)(Df)^3+\gamma\big((Df)''\cdot Df-\tfrac{5}{4}(Df)'^2\big)\Big]\cdot 1,$$

$$\phi_{h+2}(u)\to f'^{h+2}\cdot\phi_{h+2}(f(u)) + \delta f'^h\cdot Df\cdot\phi_h(f(u)),$$

$$\left(\alpha=\tfrac{1}{6}N_{000}^{224},\qquad \beta=\sqrt{\tfrac{1}{2}c}\,\tfrac{1}{6}N_{000}^{426},\qquad \gamma=\tfrac{1}{120}N_{000}^{226},\qquad \delta=\tfrac{1}{6}N_{h0h}^{022}\right).$$

We see the Schwarz derivative Df appearing term by term. It vanishes if and only if f is a Möbius transformation, in which case the transformation law reduces to eq. (1). Note that the above transformation law for ϕ_{h+2} is correct only where $f'(u)$ is positive. Beyond the singular point the nonlocal parts of the local fields will split apart due to their different complex phases, as in eq. (3). This cannot be obtained by naive integration of the infinitesimal variations.

5. The general structure of 4-point functions

Let us return to the vacuum expansion (4) and multiply it by a bilocal bra vector $\langle 0|\phi\phi$. The kernel integrations can again be performed, very similarly to the above calculations. This time the interval integrations yield Appell functions of the type F_1 (ref. [13], ch. 8) that reduce to ordinary hypergeometric functions:

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{\left(\dfrac{u_2-u_4}{u_1-u_4}\right)^{d_1-d_2}\left(\dfrac{u_1-u_3}{u_1-u_4}\right)^{d_4-d_3}}{(u_1-u_2)^{d_1+d_2}(u_3-u_4)^{d_3+d_4}}(v)$$

$$\times \sum_{h_0,\bar{h}_0} \mathrm{e}^{i\pi(h_0-\bar{h}_0)} c_{120}^{\text{prim}} c_{034}^{\text{prim}} \cdot \mathscr{F}_{h_0}(x)\bar{\mathscr{F}}_{\bar{h}_0}(\bar{x}), \quad (8)$$

where the *conformal block* $\mathscr{F}(x)$ is given by the series

$$\mathscr{F}_{h_0}(x) = x^{h_0} \sum_{n_0=0}^{\infty} \left(N_{120}^{n_1 n_2 n_0} N_{034}^{n_0 n_3 n_4}\right)(-x)^{n_0} {}_2F_1(d_2-d_1+d_0, d_3-d_4+d_0; 2d_0; x),$$

$$x := \frac{(u_1-u_2)(u_3-u_4)}{(u_1-u_3)(u_2-u_4)}, \qquad d_i = h_i + n_i. \quad (9)$$

$\bar{\mathscr{F}}(\bar{x})$ is a similar series on the v light-cone.

The finite sum in eq. (8) extends over all Hilbert space sectors which contain "intermediate states" of the 4-point function. Every term of this sum collects the contributions of all quasiprimary fields belonging to one sector. Every term in the sum (9) collects the contributions of one quasiprimary field and all of its derivatives, which would occur in a short-distance expansion as infinitely many secondary fields. If there are more than one quasiprimary fields of the same dimension d_0, n_0 should be associated with a degeneracy index to be summed over, which is indicated in eq. (9) by the brackets (NN). The series (9) converge in the "ordered region" $u_1 > u_2 > u_3 > u_4 (0 < x < 1)$ and are defined outside by appropriate analytic continuation.

Eqs. (8), (9) can be exploited in various ways. Suppose a conformal block is known from independent calculations, e.g. from Ward identities [1] or by "pseudo" Coulomb gas techniques [14]. Then the relative amplitudes for infinitely many quasiprimary fields can be read off by comparison of the power series around $x = 0$.

It can be observed that $N_{ijk}^{n_i n_j n_k} = 0$ if one of the n equals 1, indicating that quasiprimary fields of dimension one unit above the primary dimension do not exist. Similarly, in the conformal family of the identity operator ($h = \bar{h} = 0$) quasiprimary fields of dimension 1,3,5,7 do not occur, since the corresponding $N^{(n)}$ vanish identically.

The most important application of the expansions (8), (9) consists in signalling ghost state contributions to local 4-point functions. The method is illustrated, and the conditions of non-existence of states of negative norm square are evaluated recursively, in a most simple case in the next section.

6. Positivity of the energy-momentum tensor field representation

Let us specialize to the case $\phi_2 = \phi_3 = t := (\frac{1}{2}c)^{-1/2}T$ in eq. (8). $T(u)$ does not change the Hilbert space sector, and is trivial on the v light-cone. Hence $h_0 = h_1 = h_4 =: h$, and $\bar{d}_1 = \bar{d}_4 = \bar{d}_0 =: \bar{d}$. We want to consider h and c as free parameters for the moment. Introducing as in sect. 4 the coefficients $(-)^m \sqrt{\frac{1}{2}c}\, N_{h0h}^{m2n} =: N_{mn} = N_{nm}$ we get from eq. (8)

$$\left\langle \phi_{h+m,\bar{d}}(x_1) T(u_2) T(u_3) \phi_{h+n,\bar{d}}(x_4) \right\rangle$$

$$= \frac{\left(\dfrac{u_2 - u_4}{u_1 - u_4}\right)^{h+m-2} \left(\dfrac{u_1 - u_3}{u_1 - u_4}\right)^{h+n-2}}{(u_1 - u_2)^{h+m+2} (u_3 - u_4)^{h+n+2}} \frac{1}{(v_1 - v_4)^{2\bar{d}}}$$

$$\times \mathrm{e}^{-i\pi(h-\bar{h})}(-1)^m x^h \sum_{l=0}^{\infty} (N_{ml} s_l N_{ln}) x^l {}_2F_1(2 - n + l, 2 - m + l; 2l; x) . \quad (10)$$

Here we have introduced, according to our remark in sect. 3, a sign factor s_l, which equals (-1) if and only if the corresponding intermediate state created by $\phi_{h+l,\bar{d}}$ has negative norm square. It may be considered as a "metric" $s_l = \mathrm{diag}(\pm 1, \ldots, \pm 1)$ in a real vectorspace $\mathbb{R}^{D_l}$ where D_l is the number of independent quasiprimary fields of dimension $d = h + l$. The condition that no ghost fields contribute, is then expressed by $s_l = \mathbb{1}_{D_l}$ which is equivalent to the statement that all "scalar products" $(N_{ml} \cdot N_{ln}) := (N_{ml} s_l N_{ln})$ obey the rules of euclidean geometry.

Now we supplement eq. (10) by the short-distance expansion for the product of the two canonical fields $T(u)$ [1] which remains free of phase ambiguities in Minkowski quantum field theory:

$$T(u_2) \cdot T(u_3) = \frac{c/2}{(u_2 - u_3)^4} + \frac{T(u_2) + T(u_3)}{(u_2 - u_3)^2} + (\text{regular terms at } u_2 \approx u_3) . \quad (11)$$

Inserting (11) into (10) one gets the following consistency conditions:

$$\mathscr{H}_{mn}(x) := \sum_l (N_{ml} \cdot N_{ln}) x^l {}_2F_1(2-n+l, 2-m+l; 2l; x), \tag{12a}$$

$$\mathscr{H}_{mn}(x) = \mathscr{H}_{mn}(x) = x^{n+m} \mathscr{H}_{mn}(1/x), \tag{12b}$$

$$\mathscr{H}_{mn}(x) = \frac{c}{2} \delta_{mn} x^m \frac{x^2}{(1-x)^4} + N_{mn}(x^m + x^n) \frac{x}{(1-x)^2} + (\text{regular terms at } x \approx 1). \tag{12c}$$

Taking into account that $\mathscr{H}_{mn}$ has no other poles except at $x = 1, \infty$ we conclude from (12b, c)

$$\mathscr{H}_{mn}(x) = \frac{c}{2} \delta_{mn} x^m \frac{x^2}{(1-x)^4} + N_{mn}(x^m + x^n) \frac{x}{(1-x)^2} + \sum_{k=0}^{m+n} \alpha_k^{mn} x^k,$$

$$\alpha_k^{mn} = \alpha_{m+n-k}^{mn} \in \mathbb{R}. \tag{12c$'$}$$

Eqs. (12a, c$'$) allow for a recursive determination of all the numerical coefficients N_{mn} and α_k^{mn} as rational functions of the so far free parameters h and c. Every 4-point function $\mathscr{H}_{mn}(x)$ is obtained in closed form after a finite number of recurrence steps in m, n and k. Starting which the amplitudes of diagonal 3-point functions $\langle \phi_{h+n,\bar{d}} T \phi_{h+n,\bar{d}} \rangle$, which are fixed from the conformal transformation laws: $N_{nn} = h + n$, we computed for $m = n = 0$

$$(N_{01} \cdot N_{10}) = 0, \tag{13a}$$

$$(N_{02} \cdot N_{20}) = \frac{8}{2h+1}\left(h^2 - \tfrac{5}{8}h\right) + 4c \equiv \frac{8}{2h+1}(h - h_{1,2}(c))(h - h_{2,1}(c)), \tag{13b}$$

$$(N_{03} \cdot N_{30}) = \frac{6h}{(h+1)(h+2)}(h - h_{1,3}(c))(h - h_{3,1}(c)), \tag{13c}$$

$$(N_{04} \cdot N_{40}) = \frac{h(3h^3 - 76h^2 + 104h - 15)}{(h+3)(2h+3)(2h+5)} + \frac{5h(2h+1)}{(h+3)(2h+5)} c, \tag{13d}$$

where $h_{p,q}(c) = ((p(m+1) - qm)^2 - 1)/4m(m+1)$ if c is parametrized by $c = 1 - 6/m(m+1)$, $m \in \mathbb{R}$.

There is no quasiprimary field of dimension $h + 1$; in fact the only candidate is $\partial_u \phi_{h\bar{d}}$ which does not transform like a covariant tensor. If $N_{02} = N_{20}$, determined from (13b), turns out to vanish, i.e. $h = h_{1,2}(c)$ or $h = h_{2,1}(c)$, then a field of dimension $h+2$ cannot exist in the same family, since if it did exist, it would not be coupled to $\phi_{h\bar{d}}$ by the energy-momentum tensor field, and were itself primary.

Now, if $\phi_{h+2,\bar{d}}$ exists, we computed from (12a, c′) with $m=n=2$

$$(N_{21}\cdot N_{12}) = (N_{23}\cdot N_{32}) = 0\,, \tag{13e}$$

$$(N_{24}\cdot N_{42}) = \frac{32h^3+92h^2+46h+22}{(2h+1)(2h+5)} + c\,, \tag{13f}$$

and with $m=0$, $n=2$

$$(N_{01}\cdot N_{12}) = (N_{03}\cdot N_{32}) = 0\,, \tag{13g}$$

$$(N_{04}\cdot N_{42}) = \frac{72h^2}{(2h+5)^2}\left(\frac{2h(8h-5)}{(2h+1)} + c\right). \tag{13h}$$

$T(u)$ never couples quasiprimary fields with dimensions differing by one unit, since such contributions to the short-distance expansion would violate the infinitesimal Möbius transformation law (cf. sect. 4). This naturally explains eqs. (13e, g).

Positivity requires a euclidean "scalar product" $(N\cdot N)$. Since $N_{mn}=N_{nm}$ are real, (13b, c, d, f) must be non-negative numbers. Even stronger,

$$(N_{04}\cdot N_{40})(N_{24}\cdot N_{42}) - (N_{04}\cdot N_{42})^2 = \frac{128h(2h+1)}{(h+3)(2h+5)}\prod_{p\cdot q=4}\left(h-h_{p,q}(c)\right) \tag{13i}$$

must be non-negative. If (13i) turns out positive, i.e. $h \neq h_{1,4}(c), h_{4,1}(c), h_{2,2}(c)$, there are necessarily two independent quasiprimary fields of dimension $h+4$ present. We have checked such degeneracies in cases where Fock space operator expressions are available, e.g. $c=\frac{1}{2}$ (Ising) and $h=0,\frac{1}{2}$.

We recognize in (13b, c, i) $\geqslant 0$ the (first) positivity conditions of the matrix of scalar products in the Verma module V_h of the Virasoro algebra, which have been solved by Friedan, Qiu, and Shenker [2]. However, we don't have to deal with higher powers of the zeros that appear in the Kac determinant formula [3], since the contributions of all the derivatives of quasiprimary fields need not to be taken into account independently. In Verma module language, we consider only the subspace of highest-weight states $|h+n\rangle$ of the Möbius subalgebra: $L_1|h+n\rangle=0$, which are orthogonal to all "derivative states" $L_{-1}|X\rangle$.

7. Outlook

We have discussed the structure of local 4-point functions of quasiprimary fields (eq. (9)) implied by global vacuum expansions. This formula is not really new: in its euclidean version it is essentially contained in ref. [1], appendix A. We have exploited this structure for the discussion of positivity, and for the construction of closed expressions for certain 4-point functions. As far as the positivity of the representation of the energy-momentum tensor field is concerned, our technique seems to be more economic than the study of the full level-n matrix of scalar products in a Verma module. However, our results are recursive, and are given explicitly only at the first few levels. Clearly more skillful analytic techniques based on group-theoretical orthogonality properties of the respective families of hypergeometric functions are needed. The natural setting for this technical refinement seems to be Mack's "Plancherel formula" of the conformal decomposition theory [11].

In principle the same strategy applies to the study of positivity of general interpolating fields, and to the determination of local 4-point functions $\langle\phi_1\phi_2\phi_3\phi_4\rangle$. Such functions in general involve more than one conformal block (on either light-cone). It is an important feature that different conformal blocks contributing to the one local 4-point function $\langle\phi\phi\phi\phi\rangle$ are generated by analytic continuations of one "master function". This fact, which is easily observed in the Ising model, is essential for locality. In a recent study of the Ising model [6] we introduced the "exchange algebra of light-cone fields" as an abstraction of these properties of functions in terms of light-cone field operators. It provides an algebraic background for the intimate relation among different representation sectors of the energy-momentum tensor, which is responsible for the existence of modular invariant partition functions [10]. Light-cone fields may be regarded as basic constituents of the nonlocal parts of conformal fields discussed in this paper.

In order to apply our algorithm in the case of several conformal blocks contributing, it is important to control their behaviour under analytic continuation around the singular points, or, at the level of fields, to understand the correct phases of operator product expansions away from the vacuum sector. These cannot be as simple as eq. (4). Exchange algebra relations tell us that only certain linear combinations of operator products of nonlocal parts can be expected to have a definite monodromy phase and a global expansion with well-defined integral kernels. For these linear combinations the singular behaviour of a 4-point conformal block may be controlled at $x \approx 1$, and consistency as between eqs. (10) and (11) can be imposed. The solutions are expected to be at worst implicit algebraic functions.

Note added

After completion of this work we became aware of a recent paper by Furlan, Sotkov, and Todorov [15] who analyzed multiplicities of quasiprimary fields of given dimensions by explicit constructions in terms of bilocal limits. Some of their ideas and conclusions overlap with ours.

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