EXCHANGE ALGEBRA ON THE LIGHT-CONE AND ORDER/DISORDER 2n-POINT FUNCTIONS IN THE ISING FIELD THEORY

Karl-Henning REHREN and Bert SCHROER
Institut für Theorie der Elementarteilchen, Freie Universität Berlin, Arnimallee 14, D-1000 Berlin 33, Germany

Received 2 July 1987; revised manuscript received 24 July 1987

We propose an operator algebra for light-cone fields which is a generalization for $c<1$ of that for vertex operators. The algebra replaces the locality condition of conventional quantum fields. Its positive Hilbert-space representation allows for the construction of all Wightman functions of the Ising field theory ($c=\frac{1}{2}$).

Introduction. The main progress in two-dimensional conformally invariant quantum field theory has been obtained with the help of representation theory of the Virasoro algebra. On the basis of the known results [1,2] the conjecture of the existence of an algebra of "light-cone fields" which interpolate among the various irreducible representation sectors of the energy–momentum tensor (belonging to one value c of the Virasoro central extension constant) is very suggestive. The program of constructing these fields and their algebra should be a special case of the general construction of "charge carrying" fields from the representation of local algebras a la Haag et al.

As most simple light-cone fields we have in mind the vertex operator fields which however are primary only if $c=1$. The study of such fields exhibits that phases picked up in a correlator as the light-cone coordinates of two neighbouring fields are exchanged are just half the monodromy phases picked up upon analytic continuation of one coordinate along a path once surrounding another one. In conventional field theory, the exchange of operators within an n-point function is related to different $i\epsilon$ prescriptions for the relative coordinates, i.e. can be effected by some appropriate analytic continuation. On the other hand, the value of the n-point function at the exchanged configuration is related to the original one by locality. In light-cone field theory "locality" is missing. In its place we shall postulate a new type of algebra, called exchange algebra, as a means to enlarge the domain of analyticity. The "square" of this algebra must be related to the monodromy properties of the theory. In particular we are interested in a sufficiently nontrivial monodromy in order to describe the full Ising field theory in terms of different $i\epsilon$ limits of one and the same function and in order to be able to transform relatively dual fields into each other.

The algebraic structure to be described in this letter has actually been inspired from, and is clearly consistent at the four-point level. It has been tested at the six-point level with the desired result.

With some hindsight from representation theory we will argue that the light-cone exchange algebra of the Ising model can be described in terms of two primary non-hermitean fields $a(u)$, $b(u)$ both of dimension $d=\frac{1}{16}$ but with different phase behaviour under global conformal transformations. With the help of very simple Ising correlations at equal time [3–5] [#1] we then calculate the correlation functions of the light-cone fields by a process of "holomorphic factorization". From these building blocks the various 2n-point functions of the full Ising field theory may be derived. After we finished this study we received a preprint by Burkhardt and Guim [6] who calculated the Ising six-point function from the partial differential equations related to the Virasoro algebra [1]. In that approach however, the question of unitarity remains an open problem.

[#1] See especially p. 102 of ref. [4].

Exchange algebra and monodromy algebra. The Ising correlations can in principle be computed as "holomorphic square roots" of the appropriate combinations of Schwinger determinants appearing in the "doubled" theory [4,5]. However since our main purpose is to explain some new ideas and methods we shall postpone the explicit formulas. The monodromy algebra and exchange algebra relations described below are in fact read off the well known correlation functions at the four-point level and are postulated to be valid at any level.

From the analysis of Belavin, Polyakov and Zamolodchikov [1] and Dotsenko and Fateev [2] it is known that for $c=\frac{1}{2}$ one has three sectors in Hilbert space carrying representations of the energy–momentum tensor with the values for the dimensions

$$d=0, \tfrac{1}{2}, \tfrac{1}{16} .$$

We would like to introduce fields on a light-cone which interpolate between these sectors. Let $a(u)$ create the $d=\frac{1}{16}$ sector from the vacuum ($d=0$) sector. It is clear that $a(u)$ carries the dimension $d=\frac{1}{16}$. Under a global Möbius transformation $\exp(2\pi i L_0)$ generated by the conformal hamiltonian L_0, which maps a point in Minkowski space into itself but which is represented as an element of the global covering of the Möbius group, the field $a(u)$ picks up a phase

$$\exp(2\pi i L_0) a(u) |0\rangle = \exp(2\pi i \tfrac{1}{16}) a(u) |0\rangle .$$

The twofold application of this operator must give zero since there are no states with eigenvalues $\frac{1}{8}$ of L_0. Applying instead the hermitian conjugate $a^+(u_1)$:

$$a^+(u_1) a(u_2) |0\rangle \in \text{vacuum sector}$$

leads back to the vacuum sector.

In order to obtain the $d=\frac{1}{2}$ sector from the $d=\frac{1}{16}$ sector we therefore need another field $b(u)$ which under the global Möbius transformation (see above) picks up a phase $\exp(2\pi i \frac{7}{16})$:

$$b(u_1) a(u_2) |0\rangle \in d=\tfrac{1}{2} \text{ sector} .$$

We do not want this field to generate states with dimension $\frac{7}{16}$ from the vacuum, hence

$$b(u)|0\rangle = 0, \quad a^+(u)|0\rangle = 0 .$$

This would be impossible in local quantum field theory, but the light-cone fields are not local fields. Locality will be replaced by another structure called exchange algebra to be discussed in the following.

From the above we see that there are only two non-vanishing four-point correlators that can contribute to four-point functions of the Ising fields μ and σ:

$$\langle a^+(u_1) a(u_2) a^+(u_3) a(u_4) \rangle, \quad \langle a^+(u_1) b^+(u_2) b(u_3) a(u_4) \rangle . \tag{1}$$

By definition we choose these functions such that their cuts fall outside the ordered region

$$u_1 > u_2 > u_3 > u_4 .$$

Let us split off a vertex correlation factor proper to fields of dimension $\frac{1}{16}$:

$$V(u_1, u_2, u_3, u_4) = \langle :\exp[i\varphi(u_1)]: :\exp[-i\varphi(u_2)]: :\exp[i\varphi(u_3)]: :\exp[-i\varphi(u_4)]: \rangle ,$$

with $\langle \varphi(u)\varphi(u') \rangle = -\frac{1}{8}\log(u-u')$, and write

$$\langle a^+(u_1) a(u_2) a^+(u_3) a(u_4) \rangle = V(u_s) f(x) , \quad \langle a^+(u_1) b^+(u_2) b(u_3) a(u_4) \rangle = V(u_s) g(x) ,$$

$$x = (u_1 - u_2)(u_3 - u_4)/(u_1 - u_3)(u_2 - u_4) , \tag{2}$$

where the "reduced" functions f and g can have cuts extending from $+1$ to $+\infty$ and from $-\infty$ to 0.

We expect that the correlators evaluated at a configuration where the coordinates of two of the fields are interchanged (i.e. outside the ordered region), are related to the original ones. This will be expressed by the following "exchange algebra" of the light-cone fields:

$$\begin{pmatrix} a(u')a^+(u) \\ b^+(u')b(u) \end{pmatrix} = \exp(i\pi\tfrac{1}{8}) A_E \begin{pmatrix} a(u)a^+(u') \\ b^+(u)b(u') \end{pmatrix}, \tag{3a}$$

$$a^+(u')a(u) = \exp(i\pi\tfrac{1}{8})\exp(-i\varphi_1)a^+(u)a(u'), \quad b(u')a(u) = \exp(i\pi\tfrac{1}{8})\exp(-i\varphi_2)b(u)a(u'), \tag{3b}$$

$$b(u')b^+(u) = \exp(i\pi\tfrac{1}{8})\exp(-i\varphi_3)b(u)b^+(u') \quad \text{if } u > u'. \tag{3c}$$

The vertex phase $\exp(2\pi i d) = \exp(+i\pi\frac{1}{8})$ is the complex phase of the vertex correlation $V = \langle ... :\exp[\pm i\varphi(u')]: \times :\exp[\mp i\varphi(u)]: ... \rangle$ at $u > u'$. It has been split off in (3) such that A_E and $\exp(-i\varphi_i)$ describe the new features beyond the algebra of vertex operators. The $\exp(2\pi i L_0)$ transformation law allows no further mixing of field products than the one described by a 2×2 matrix A_E, satisfying $A_E^{-1} = A_E^*$ for self-consistency of (3a).

By definition, a correlator, say $\langle ...a(u')a^+(u)...\rangle$ with $u' < u$ is obtained from $\langle ...a(u)a^+(u')...\rangle$ in the ordered region by an analytic continuation in u and u' exchanging the two and approaching the "new" u' from below and the "new" u from above the cut: in other words, it pulls the cut originally extending from u to the "old" u' position and simultaneously moves the point u' above the cut to the "old" u position. Performing this analytic exchange twice amounts to the analytic continuation in the u' variable clockwise once around u. Correspondingly, the algebra (3) implies for the "reduced" functions f and g

$$x^{1/4}\begin{pmatrix} f_+(1/x) \\ g_+(1/x) \end{pmatrix} = A_E \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \tag{4a}$$

$$(1-x)^{1/4} f_-(-x/(1-x)) = \exp(-i\varphi_1) f(x), \quad (1-x)^{1/4} g_-(-x/(1-x)) = \exp(-i\varphi_2) g(x) \quad \text{if } 0 < x < 1, \tag{4b}$$

where f_+, g_+ (f_-, g_-) denote the values above (below) the respective cuts. (3c) cannot be tested at the four-point level.

The exchange phases $\varphi_{1,2}$ and A_E can be computed from the well-known four-point functions, φ_3 was fitted at the six-point level. They then fix the exchange and monodromy behaviour of all higher $2n$-point functions. The result is

$$\varphi_1 = 0, \quad \varphi_2 = \tfrac{1}{2}\pi, \quad \varphi_3 = 0, \quad A_E = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp(-i\pi\frac{1}{4}) & \exp(i\pi\frac{1}{4}) \\ \exp(i\pi\frac{1}{4}) & \exp(-i\pi\frac{1}{4}) \end{pmatrix}. \tag{5}$$

From this we deduce the monodromy behaviour of f, g as u_i surrounds u_{i+1} ($u_i \sim u_{i+1}$):

$$\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \to A_E^2 \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} g(x) \\ f(x) \end{pmatrix}, \quad \text{as } x \sim 1 \ (u_2 \sim u_3), \tag{6a}$$

$$f(x) \to \exp(-2i\varphi_1) f(x) = f(x), \quad \text{as } x \sim 0 \ (u_1 \sim u_2 \text{ or } u_3 \sim u_4),$$

$$g(x) \to \exp(-2i\varphi_2) g(x) = -g(x), \quad \text{as } x \sim 0 \ (u_1 \sim u_2 \text{ or } u_3 \sim u_4). \tag{6b}$$

It is very satisfactory that monodromy connects f and g in this simple way. The monodromy is of order 2 which is most desirable for the Ising model.

The algebraic setting presented in this section can be easily generalized to other field theoretical models with conformal invariance and $c = 1 - 6/m(m+1)$.

Construction of Ising 2n-point functions. We shall from now on consider the exchange algebra relations (3) (5) as the fundamental input for the construction of Ising $2n$-point functions.

With the monodromy properties (6) of the "reduced" functions f and g at hand, it is almost straightforward to write down the local and dual four-point correlations

$$\langle\sigma(\boldsymbol{x}_1)\sigma(\boldsymbol{x}_2)\sigma(\boldsymbol{x}_3)\sigma(\boldsymbol{x}_4)\rangle=\langle\mu(\boldsymbol{x}_1)\mu(\boldsymbol{x}_2)\mu(\boldsymbol{x}_3)\mu(\boldsymbol{x}_4)\rangle=V(u_s)V(v_s)[f(x)f(\bar{x})+g(x)g(\bar{x})]\,,$$

$$\langle\sigma\sigma\mu\mu\rangle=\langle\mu\mu\sigma\sigma\rangle=V\bar{V}(f\bar{f}-g\bar{g}),$$

$$\langle\sigma\mu\mu\sigma\rangle=\langle\mu\sigma\sigma\mu\rangle=V\bar{V}(f\bar{g}+g\bar{f}),\quad \langle\sigma\mu\sigma\mu\rangle=-\langle\mu\sigma\mu\sigma\rangle=V\bar{V}\mathrm{i}(f\bar{g}-g\bar{f})\,, \tag{7}$$

with

$$\boldsymbol{x}=\begin{pmatrix}x^0\\x^1\end{pmatrix},\quad u=x^0+x^1\,,\quad v=-x^0+x^1\,,\quad x=(u_1-u_2)(u_3-u_4)/(u_1-u_3)(u_2-u_4)\,,$$

and barred quantities referring to the other light-cone. These combinations are unique to satisfy the rule that whenever the coordinate of one field is moved once around the coordinate of another field local (dual) with respect to the former (which amounts to the simultaneous monodromy operation for both light-cone variables), the correlation function picks up the factor $+1(-1)$. The function $f(x)$ can then be computed from the correlators (7) evaluated at equal time, i.e. $x=\bar{x}$, where they simplify considerably. The formulae obtained by Kadanoff and Ceva [3] on the lattice can be rederived in field theory in terms of the "doubled" theory [4,5]. The result is

$$f^2(x)+g^2(x)=V^{-2}\langle\sigma\sigma\sigma\sigma\rangle_{\text{equal time}}=1\,,\quad f^2(x)-g^2(x)=V^{-2}\langle\sigma\sigma\mu\mu\rangle_{\text{equal time}}=\sqrt{1-x}\,,$$

hence

$$f(x)=(1/\sqrt{2})\sqrt{1+\sqrt{1-x}},\quad g(x)=(1/\sqrt{2})\sqrt{1-\sqrt{1-x}}\,. \tag{8}$$

The exchange properties (4) of these functions are easily verified.

We want to emphasize that we would have liked better to be able to determine the four-point correlators f and g of the light-cone fields as the unique representations of the exchange algebra that satisfy the appropriate positivity requirements. The study of an adequate algorithm for this problem is in progress. For the moment we had to rely on the independent knowledge of equal-time four-point functions. Still there is the great advantage that these are extremely simple functions, and only two non-vanishing correlations are needed for the determination of all mixed four-point functions (7) of local and dual fields of the Ising model.

This technical simplification allows us to compute the full $2n$-point functions of local and dual Ising fields: Instead of (2) there are 2^{n-1} functions $f_1=V^{-1}\langle a^+a...a^+a\rangle, f_2, ...$, the monodromy behaviour of which ($f_i\to\pm f_i$, or $f_i\leftrightarrow f_j$) is immediately read off as in (6). The construction of the mixed $2n$-point functions of $\phi_s=\mu$, σ in terms of f_i as in (7) is a rather simple combinatorial problem, in particular

$$\langle\sigma(\boldsymbol{x}_1)...\sigma(\boldsymbol{x}_{2n})\rangle=V(u_s)V(v_s)\sum_{i=1}^{2^{n-1}}f_i(u_s)f_i(v_s)\,, \tag{9}$$

and

$$f_1(u_s)f_1(v_s)=2^{1-n}[V(u_s)V(v_s)]^{-1}\sum(\text{all different functions }\langle\phi_1...\phi_{2n}\rangle\text{ such that }\phi_{2p-1}=\phi_{2p})\,. \tag{10}$$

The $2n$-point functions appearing in this sum are computed at equal time in the ordered region $u_1=v_1>u_2=v_2>...>u_{2n}=v_{2n}$:

$$\langle\phi_1(\boldsymbol{x}_1)...\phi_{2n}(\boldsymbol{x}_{2n})\rangle_{\text{equal time}}=V^2\prod_{\substack{p\\(\phi_{2p-1}=\phi_{2p}=\sigma)}}\;\prod_{\substack{q\\(\phi_{2q-1}=\phi_{2q}=\mu)}}\sqrt{1-x_{pq}}\,,$$

$$x_{pq}=x_{qp}=(u_{2p-1}-u_{2p})(u_{2q-1}-u_{2q})/(u_{2p-1}-u_{2q})(u_{2q-1}-u_{2p})\,,\quad p,q=1,...,n\,. \tag{11}$$

Hence

$$f_1(u_s)=\left(2^{-n}\sum_{\substack{I_1\cup I_2=\{1,\dots,n\}\\ I_1\cap I_2=\emptyset}}\prod_{p\in I_1}\prod_{q\in I_2}\sqrt{1-x_{pq}}\right)^{1/2}. \tag{12}$$

(In this sum every terms appears twice due to the symmetry $I_1 \leftrightarrow I_2$.)

The other functions f_i, $i=2, \ldots, 2^{n-1}$, are obtained from f_1 by all possible sign flips $\sqrt{1-x_{pp+1}} \to -\sqrt{1-x_{pp+1}}$, $p=1, \ldots, (n-1)$, since all monodromies are induced from the "diagonal" ones: $u_{2p-1} \sim u_{2p}$, $x_{pq} \sim 0$, $f_i \to \pm f_i$, and the "offdiagonal" ones: $u_{2p} \sim u_{2p+1}$, $x_{pp+1} \sim 1$, $f_i \leftrightarrow f_j$. With

$$V(u_s)=\left(\prod_{p=1}^{n}(u_{2p-1}-u_{2p})\cdot\prod_{1\leqslant p<q\leqslant n}(1-x_{pq})\right)^{-1/8},$$

we finally get the complete local $2n$-point function

$$\langle\sigma(\boldsymbol{x}_1)\ldots\sigma(\boldsymbol{x}_{2n})\rangle=V(u_s)V(v_s)$$

$$\times\sum_{\substack{\text{different signs, simultaneously chosen}\\ \text{for } \sqrt{1-x_{pp+1}} \text{ and } \sqrt{1-\bar{x}_{pp+1}}}}\left(2^{-n}\sum_{I_1 I_2}\prod_{\substack{p\in I_1\\ q\in I_2}}(\pm)\sqrt{1-x_{pq}}\right)^{1/2}\cdot\left(2^{-n}\sum_{I_1 I_2}\prod_{\substack{p\in I_1\\ q\in I_2}}(\pm)\sqrt{1-\bar{x}_{pq}}\right), \tag{13}$$

and appropriate combinations of the same functions for the mixed $2n$-point functions.

We observe that the monodromy operation in one light-cone only, e.g. moving u_s around u_{s+1} has the effect of changing both ϕ_s and ϕ_{s+1} into their dual fields ($\sigma \leftrightarrow \mu$) and leaving all other fields in the $2n$-point function unchanged. This indicates the existence of some operator transformation connecting the dual fields σ and μ with each other – at least in bilocal combinations.

References

[1] A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333.
[2] V.S. Dotsenko and V.A. Fateev, Nucl. Phys. B 240 (1984) 312; B 251 (1984) 514.
[3] L. Kadanoff and H. Ceva, Phys. Rev. B 3 (1971) 3918.
[4] B. Schroer and T.T. Truong, Nucl. Phys. B 144 (1978) 80.
[5] A. Luther and I. Peschel, Phys. Rev. B 12 (1975) 3908.
[6] Th. Burkhardt and I. Guim, Temple University preprint (1987).