CONSTRAINTS ON THE CURRENTS IN PRINCIPAL MODELS GIVING RISE TO INTEGRABLE NONLINEAR SYSTEMS

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Quadratic constraints on the currents in the principal SU(2n) model stand at the origin of an extended reduction mechanism. We derive local conservation laws both for the reduced model and for a class of solutions of the principal SU(2n) model.

Introduction. The method of reduction is a powerful concept within the discussion of integrable two-dimensional theories, since it elucidates relations between σ-models and certain other nonlinear systems. Let us briefly review the basic ideas:

We know [1] how to relate a parametric linear system to the principal G-model on the group G. The restriction of the principal fields to some symmetric space G/H [2,3] provides the linear system with additional structure. We can look for a gauge equivalent linear system, which as expression of this structure takes a particularly simple form [1,4]. The nonlinear compatibility equation now involving a "minimal" number of fields defines the reduced version of the G/H σ-model. Since a linear problem is automatically related to it, the time evolution of the reduced model can be solved following the inverse scattering pattern.

In part B of this letter, the following observation will enlarge the class of integrable models which can be obtained by the reduction of σ-models: Quadratic constraints on the *currents* of the principal model as well as those on the *fields* itself can give rise to suitable structure of the linear system, allowing for reduction. This is demonstrated by an example (G = SU(2n)).

In part A we shall firstly discuss a weaker constraint on the principal SU(2n) currents and find the local conservation laws of the theory. These carry over to the reduced model of part B.

A. Conservation laws. The principal SU(m) model is defined by the Lagrangian density $\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\, S_\xi S^+ S_\eta S^+$, $S \in \mathrm{SU}(m)$ [‡1]. The equation of motion is $2S_{\xi\eta} = S_\xi S^+ S_\eta + S_\eta S^+ S_\xi$. Ogielski et al. [5] have found the local conservation laws $\mathrm{Tr}(G_{N+2}S_\xi^+ + S_\xi G_{N+2}^+)_\eta + \mathrm{Tr}(G_N S_\eta^+ + S_\eta G_N^+)_\xi = 0$, together with a recursion formula for G_N, which, however, could not be solved explicitly. This task will be carried out when we concentrate on solutions satisfying

$$S_\xi S_\xi^+ = S_\eta S_\eta^+ = \mathbf{1}\,. \tag{1}$$

Note, that these properties are reproduced by virtue of the evolution equation of the principal field, if they are once (in the initial data) realized.

Eq. (1) means that $S_\xi S^+$ is an element of SU(m) and su(m) at the same time. The eigenvalues of SU(m) matrices are of modulus 1, those of su(m) matrices are imaginary and sum up to zero. Thus (1) is only meaningful for $m = 2n$.

Actually it suffices to require $S_\xi S_\xi^+ = \mathrm{real}\cdot\mathbf{1}$, $S_\eta S_\eta^+ = \mathrm{real}\cdot\mathbf{1}$. The normalization can then be performed by a rescaling of coordiantes.

Now the solution of the recursion formula of ref. [5] is

$$G_0 = S\,,\quad G_1 = 2S_\xi\,,$$

[‡1] Here and in the sequel, the subscripts ξ and η denote partial differentiation with respect to light cone coordinates $\xi = \frac{1}{2}(t+x)$, $\eta = \frac{1}{2}(t-x)$; $^+$ means Hermitean conjugation.

$$G_N = \epsilon_{N-1} S_\xi - \tfrac{1}{4} \sum_{\nu=2}^{N-1} G_\nu G^+_{N+1-\nu} S_\xi + \tfrac{1}{2} \sum_{\nu=1}^{N-1} G_{\nu\xi} G^+_{N-\nu} S_\xi \,, \quad N > 1 \,,$$

where $\epsilon_N = 0$ for odd N, $\epsilon_N = (-1)^{N/2}$ for even N.

We give the first two nontrivial conservation laws:

$$\mathrm{Tr}(S_{\xi\xi} S^+_{\xi\xi})_\eta - \mathrm{Tr}(S_\xi S^+_\eta + S_\eta S^+_\xi)_\xi = 0 \,,$$

$$\mathrm{Tr}(S_{\xi\xi\xi} S^+_{\xi\xi\xi} - \tfrac{5}{4} S_{\xi\xi} S^+_{\xi\xi} S_{\xi\xi} S^+_{\xi\xi})_\eta + \mathrm{Tr}(\tfrac{1}{2} S_{\xi\xi} S^+_{\xi\xi} (S_\xi S^+_\eta + S_\eta S^+_\xi))_\xi = 0 \,.$$

B. Reduction. The linear system associated to the SU($2n$) model is [1]

$$\Phi_\xi = \mathrm{U}\Phi, \qquad \Phi_\eta = \mathrm{V}\Phi \,,$$

$$\mathrm{U}(\zeta) \equiv \zeta \mathrm{A} + \mathrm{C} := -\frac{\zeta - 1}{2} S_\xi S^+ \,,$$

$$\mathrm{V}(\zeta) \equiv \zeta^{-1} \mathrm{B} + \mathrm{D} := -\frac{\zeta^{-1} - 1}{2} S_\eta S^+ \,.$$

Let us now intensify the constraint (1) by the additional requirement

$$S_\xi S^+_\eta + S_\eta S^+_\xi = \mathrm{real} \cdot \mathbf{1} \,. \qquad (2)$$

This constraint is not automatically reproduced by the field equation; instead, it implies a series of further relations like $S_{\xi\xi} S^+_\eta + S_\eta S^+_{\xi\xi} = \mathrm{real} \cdot \mathbf{1}$, which contain additional restrictions for the initial data. Then we can find two SU($2n$) gauge transformations $\Phi \to h^+_i \Phi$, $i = 1, 2$, such that the coefficients of $\mathrm{U}_i = \mathrm{h}_i \mathrm{U} \mathrm{h}^+_i + \mathrm{h}_{i\xi} \mathrm{h}^+$ and $\mathrm{V}_i = \mathrm{h}_i \mathrm{V} \mathrm{h}^+_i + \mathrm{h}_{i\eta} \mathrm{h}^+$ with respect to powers of ζ become

$$\mathrm{A}_1 = -\frac{i}{2}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

$$\mathrm{B}_1 = \frac{\mathrm{i}}{2}\begin{pmatrix} -\sqrt{\mathbf{1} - \varphi\varphi^+} & \varphi^+ \\ \varphi & \sqrt{\mathbf{1} - \varphi\varphi^+} \end{pmatrix},$$

$$\mathrm{C}_1 = \frac{1}{2}\begin{pmatrix} 0 & \dfrac{\varphi^+_\xi}{\sqrt{\mathbf{1} - \varphi\varphi^+}} \\ -\dfrac{\varphi_\xi}{\sqrt{\mathbf{1} - \varphi\varphi^+}} & 0 \end{pmatrix}, \qquad \mathrm{D}_1 = 0 \,,$$

where φ is a $n \times n$ matrix satisfying $\varphi\varphi^+ = \mathrm{real} \cdot \mathbf{1}$, and, respectively,

$$\mathrm{A}_2 = -\frac{\mathrm{i}}{2}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

$$\mathrm{B}_2 = \frac{\mathrm{i}}{2}\begin{pmatrix} 2\vartheta\vartheta^+ - \mathbf{1} & 2\sqrt{\vartheta\vartheta^+(\mathbf{1} - \vartheta\vartheta^+)} \\ 2\sqrt{\vartheta\vartheta^+(\mathbf{1} - \vartheta\vartheta^+)} & \mathbf{1} - 2\vartheta\vartheta^+ \end{pmatrix},$$

$$\mathrm{C}_2 = \frac{1}{2}\begin{pmatrix} \dfrac{\mathbf{1} - 2\vartheta\vartheta^+}{\vartheta\vartheta^+(\mathbf{1} - \vartheta\vartheta^+)} (\vartheta_\xi \vartheta^+ - \vartheta\vartheta^+_\xi) & \dfrac{2\vartheta\vartheta^+_\xi}{\sqrt{\vartheta\vartheta^+(\mathbf{1} - \vartheta\vartheta^+)}} \\ -\dfrac{2\vartheta_\xi\vartheta^+}{\sqrt{\vartheta\vartheta^+(\mathbf{1} - \vartheta\vartheta^+)}} & 0 \end{pmatrix}$$

$$\mathrm{D}_2 = \frac{1}{2}\begin{pmatrix} \dfrac{\vartheta_\eta \vartheta^+ - \vartheta\vartheta^+_\eta}{\vartheta\vartheta^+(\mathbf{1} - \vartheta\vartheta^+)} & 0 \\ 0 & 0 \end{pmatrix},$$

where ϑ is a $n \times n$ matrix satisfying $\vartheta\vartheta^+ = \mathrm{real} \cdot \mathbf{1}$, too. The compatibility $\mathrm{U}_{i\eta} - \mathrm{V}_{i\xi} + [\mathrm{U}_i, \mathrm{V}_i] = 0$ gives the nonlinear differential equations

$$\varphi_{\xi\eta} + \frac{1}{2} \frac{(\varphi\varphi^+)_\eta}{\mathbf{1} - \varphi\varphi^+} \varphi_\xi + \sqrt{\mathbf{1} - \varphi\varphi^+}\, \varphi = 0 \,, \; \varphi\varphi^+ = \mathrm{real} \cdot \mathbf{1} \,,$$

$$\vartheta_{\xi\eta} + \frac{\vartheta_\xi \vartheta^+ \vartheta_\eta}{\mathbf{1} - \vartheta\vartheta^+} + (\mathbf{1} - \vartheta\vartheta^+)\,\vartheta = 0 \,, \; \vartheta\vartheta^+ = \mathrm{real} \cdot \mathbf{1} . \qquad (3)$$

They define equivalently the reduced model of our interest. ϑ and φ are related by the invertible transformation

$$\vartheta_\xi \vartheta^+ = \frac{1}{4\sqrt{\mathbf{1} - \varphi\varphi^+}} \varphi_\xi \varphi^+ \,,$$

$$\vartheta_\eta \vartheta^+ = \frac{\mathbf{1} + \sqrt{\mathbf{1} - \varphi\varphi^+}}{4} \left(\frac{\varphi}{\mathbf{1} + \sqrt{\mathbf{1} - \varphi\varphi^+}} \right)_\eta \varphi^+ \,,$$

$$\vartheta\vartheta^+ = \tfrac{1}{2}(\mathbf{1} - \sqrt{\mathbf{1} - \varphi\varphi^+}) \,.$$

Under the *special* assumption $\varphi = \Sigma_{i=1}^{N-2} \varphi_i \Gamma^i$, where Γ^i are constant matrices with the algebra $\Gamma^i \Gamma^{j+} + \Gamma^j \Gamma^{i+} = \Gamma^{i+} \Gamma^j + \Gamma^{j+} \Gamma^i = 2\delta^{ij}$, the system (3) becomes equivalent to the reduced O(N) (more precisely: O(N)/O($N-1$)) σ-model [6].

It is not likely that the constraints (1) and (2) defining the *full* system (3) can be resolved solely by some geometric assumptions over the principal field, too. However, this claim remains to be investigated exactly.

Let us now prove, that (3) is really equivalent to the principal SU(2n) model constrained by (1) and (2): The smallest symmetric Lie algebra $\mathcal{G} = \mathrm{f} \oplus \mathrm{k}$, in which U_i, V_i can be embedded such that [‡2] $A, B \in \mathrm{k}$; $C, D \in \mathrm{f}$; $[\mathrm{f},\mathrm{f}] \subset \mathrm{f}$; $[\mathrm{f},\mathrm{k}] \subset \mathrm{k}$; $[\mathrm{k},\mathrm{k}] \subset \mathrm{f}$, is $\mathcal{G} = \mathrm{su}(2n) \oplus \mathrm{su}(2n)$. f is its diagonal, k the off diagonal. U and V are embedded by

$$U \to \hat{U} = \begin{pmatrix} \zeta A + C & 0 \\ 0 & -\zeta A + C \end{pmatrix},$$

$$V \to \hat{V} = \begin{pmatrix} \zeta^{-1}B + D & 0 \\ 0 & -\zeta^{-1}B + D \end{pmatrix} \in \mathcal{G}.$$

The symmetric space is $SU(2n) \otimes SU(2n)/\Delta(SU(2n)) \approx SU(2n)$, the σ-model is the principal SU(2n) model [3].

Let $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$ be a solution of the linear problem $g_\xi = \hat{U}(\zeta = 1)g$, $g_\eta = \hat{V}(\zeta = 1)g$. Then [4] $S = g_2^+ g_1$ is the principal field. In particular this implies $S_\xi = 2g_2^+ A g_1$, $S_\eta = 2g_2^+ B g_1$, and therefore (1) and (2).

Ogielski's et al. [5] Bäcklund transformation of the principal model conserves constraint (1) but destroys constraint (2). Hence it cannot be used as a Bäcklund transformation for the reduced model (3). Nevertheless, the series of local conservation laws generated by it is valid in the reduced model too. Inserting $S = g_2^+ g_1$ the conservation laws of part A are easily expressed in terms of ϑ and φ; for illustration:

$$\mathrm{Tr}((1 - \vartheta\vartheta^+)^{-1} \vartheta_\xi \vartheta_\xi^+)_\eta + \mathrm{Tr}(\vartheta\vartheta^+)_\xi = 0 ,$$

$$\mathrm{Tr}((1 - \varphi\varphi^+)^{-1} \varphi_\xi \varphi_\xi^+)_\eta + \mathrm{Tr}(-2(1 - \varphi\varphi^+)^{1/2})_\xi = 0 .$$

These laws were already found in [6].

Conclusion. We showed the equivalence of the principal SU(2n) model subject to constraints on the currents and the integrable nonlinear system (3). This is actually an example how the concept of reduction, up to now based on field constraints, can be generalized.

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‡2 Omitting the indices $i = 1, 2$ in the following should not lead to confusion.